THE INFINITE DIMENSIONAL EVANS FUNCTION

YURI LATUSHKIN AND ALIN POGAN

Abstract. We introduce generalized operator valued Jost solutions of first order ill-posed differential equations on Hilbert spaces. We then construct an infinite dimensional Evans function for abstract differential equations as a 2–modified Fredholm determinant of the operator obtained by adding the values at zero of the generalized operator valued Jost solutions. Next, we prove a formula that connects the 2–modified Evans determinant and the 2–modified determinant related to the Birman–Schwinger type operator associated to the ill-posed equation. Using this formula, we construct a holomorphic infinite dimensional Evans function for second order differential operators on infinite cylinders whose zeros are the eigenvalues of the differential operators.

1. Introduction

The finite dimensional Evans function is a widely used tool for detecting eigenvalues of the differential operators that appear after linearizing partial differential equations about such special solutions as steady states, traveling waves, etc. There is a big fascinating literature on the subject, we refer to [1, 3, 15, 23, 28, 29, 38, 39] and more literature cited in review [33]. The construction of the *infinite dimensional* Evans function in a general setting is a longstanding open problem, although there is an extensive literature on this topic mainly related to problems on infinite cylinders, see [10, 11, 17, 27] and the literature therein.

In this paper we introduce the Evans function for fairly general infinite dimensional systems using the (modified) Fredholm determinants of integral operators of the Birman–Schwinger type, continuing the line of analysis in [14] and [7, 9, 17, 24, 26]. In particular, we show that the Evans function is a holomorphic function of a spectral parameter, and that the zeros of the Evans function are the eigenvalues of the related differential operator.

To explain the main new ideas of the current work, we briefly review the line of arguments in [14]. The Evans function $E = E(\lambda)$ is associated in [14] to a pair of unperturbed and perturbed differential equations,

$$u'(t) = Au(t), \quad u'(t) = (A + B(t))u(t), \quad t \in \mathbb{R}. \tag{1.1}$$

Following [14], we assume for the moment that $A = A(\lambda)$ and $B(t)$ are $(d \times d)$ matrices, $u(t) \in \mathbb{C}^d$ and $\lambda \in \Omega \subset \mathbb{C}$ is a spectral parameter. Let us assume that $A = A(\lambda)$ has no spectrum on the imaginary axis, and denote by $\mathcal{Q}_+ = \mathcal{Q}_+(\lambda)$,

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respectively, $\mathcal{Q}_- = \mathcal{Q}_-(\lambda)$ the spectral projections of $A(\lambda)$ corresponding the part of the spectrum $\sigma(A(\lambda))$ located in the left, respectively, in the right half–plane. Denoting the real parts of the eigenvalues of A by $\varkappa_j$ and numbering them so that $\dots < \varkappa_{-2} < \varkappa_{-1} < 0 < \varkappa_1 < \varkappa_2 < \dots$, and denoting by Q_j the spectral projection corresponding to the eigenvalues of A whose real parts are equal to $\varkappa_j$, we obtain finer decompositions $\mathcal{Q}_+ = \sum_{j<0} Q_j$ and $\mathcal{Q}_- = \sum_{j>0} Q_j$ invariant with respect to the semigroups $T_\pm(t) = e^{\pm t A_\pm}$, $t \geq 0$, where $A_\pm = \pm A_{|\,\mathrm{im}\,\mathcal{Q}_\pm}$. Since $\operatorname{Re}\sigma(A_{|\,\mathrm{im}\,\mathcal{Q}_+}) < 0$ and $\operatorname{Re}\sigma(-A_{|\,\mathrm{im}\,\mathcal{Q}_-}) < 0$, the operator $A = A_+ \oplus (-A_-)$ generates a stable bi–semigroup and $T_\pm(\cdot)Q_j$ are the matrix valued solutions of the unperturbed matrix equation $Y'(t) = AY(t)$, $t \in \mathbb{R}_\pm$, that decay to zero at the exponential rate $\varkappa_j$ as $t \to \pm\infty$ for $\pm j < 0$. The generalized matrix valued Jost solutions $Y_\pm^{(j)}(\cdot)$ are defined in [14] as the matrix solutions of the perturbed matrix equation $Y'(t) = (A + B(t))Y(t)$, $t \in \mathbb{R}_\pm$ that decay to zero exponentially as $t \to \pm\infty$ for $\pm j < 0$ so that the difference $Y_\pm^{(j)}(t) - T_\pm(t)Q_j$ decays to zero as $t \to \pm\infty$ faster than $e^{\varkappa_j t}$, $\pm j < 0$. As shown in [14] under natural assumptions on the perturbation, the generalized matrix valued Jost solutions exist, and then one can define the *Evans determinant*, $E = \det(\mathcal{Y}_+ + \mathcal{Y}_-)$, where $\mathcal{Y}_\pm = \sum_{\pm j<0} Y_\pm^{(j)}(0)$ are computed via the values of the generalized matrix valued Jost solutions at $t = 0$. The main result of [14] is an explicit formula,

$$\det\nolimits_{2,L^2}(I - \mathcal{K}) = e^{\Theta} \det(\mathcal{Y}_+ + \mathcal{Y}_-), \tag{1.2}$$

relating the Evans function $E = E(\lambda)$ and the 2–modified Fredholm determinant in $L^2 = L^2(\mathbb{R}, \mathbb{C}^d)$ for a Birman–Schwinger type integral operator, $\mathcal{K}$, whose main property is that $1 \in \sigma(\mathcal{K})$ if and only if the perturbed equation in (1.1) has a solution exponentially decaying to zero at both $+\infty$ and $-\infty$. The constant Θ in (1.2) is explicitly computed by

$$\Theta = \int_0^\infty \operatorname{tr}(\mathcal{Q}_+ B(t))\, dt - \int_{-\infty}^0 \operatorname{tr}(\mathcal{Q}_+ B(t))\, dt. \tag{1.3}$$

It turns out that $\mathcal{K} = \mathcal{K}(\lambda)$ is an analytic family of Hilbert–Schmidt operators, provided $A(\lambda)$ is analytic, and then (1.3) immediately yields the analyticity of the Evans function $E = E(\lambda)$.

The main objective of the current work is to show how the approach of [14] can be naturally developed for *infinite dimensional* equations (1.1). We did not strive for the most general or optimal assumptions, but just wanted to open new venues through which the infinite dimensional Evans function can be explored. Specifically, in this paper we work on the following:

- Given ill–posed first order infinite dimensional differential equations of the form (1.1) on a Hilbert space $\mathcal{X}$, we construct and analyze the associated Birman–Schwinger operator $\mathcal{K}$ on $L^2(\mathbb{R}, \mathcal{X})$. In particular, we are interested in finding the conditions that guarantee that the operator $\mathcal{K}$ belongs to the Schatten–von Neumann ideals. In addition, we show that the operator $\mathcal{K}$ can be approximated in the right trace ideal norm by a sequence of Birman–Schwinger type integral operators with semi–separable kernels associated to the Galerkin approximation of the infinite dimensional equations (1.1).
- We prove the existence of the generalized operator valued forward/backward Jost solutions of the perturbed infinite dimensional equation in (1.1), and construct the operators $\mathcal{Y}_+$ and $\mathcal{Y}_-$ as the infinite sums of the values of the Jost solutions

at zero. Similarly to the finite dimensional case, these solutions are asymptotic to the reference operator valued solutions of the unperturbed infinite dimensional equation in (1.1). Next, we show that $\mathcal{Y}_+ + \mathcal{Y}_- - I$ is a Hilbert–Schmidt operator which allows us to define the 2-modified Evans determinant by the formula $E = \det_{2,\mathcal{X}}(\mathcal{Y}_+ + \mathcal{Y}_-)$. We also prove that the Jost solutions and the Evans determinant for the infinite dimensional equations (1.1) can be approximated by a sequence of the Jost solutions and the 2–modified Evans determinants associated to the finite dimensional Galerkin approximation of equations (1.1).

• We establish connections between the 2–modified determinant of the Birman–Schwinger operator $\mathcal{K}$ and the 2–modified infinite dimensional Evans determinant by proving that

$$\det_{2,L^2(\mathbb{R},\mathcal{X})}(I - \mathcal{K}) = e^{\Xi} \det_{2,\mathcal{X}}(\mathcal{Y}_+ + \mathcal{Y}_-), \tag{1.4}$$

where Ξ can be expressed via the trace–like formula

$$\Xi = \operatorname{tr}\left[\operatorname{diag}\left(\mathcal{Y}_+ + \mathcal{Y}_- - I + \int_{\mathbb{R}_+} \mathcal{Q}_+ B(t)\,dt - \int_{\mathbb{R}_-} \mathcal{Q}_- B(t)\,dt\right)\right]. \tag{1.5}$$

This result generalizes the results from [14] and [17], and yields a new formula specific for the infinite dimensional case at hand.

• Given an eigenvalue problem for a second order differential operator $\mathcal{L}$, we use the Evans determinant to construct a holomorphic Evans function of the spectral parameter whose zeros correspond to the eigenvalues of $\mathcal{L}$. The formula (1.4) is a crucial ingredient in the proof of analyticity of the Evans function.

• Finally, we mention an important issue that is *not* addressed in this paper in full generality: We do not construct the Jost solutions under the natural general assumption $\int_{\mathbb{R}} \|B(t)\|_2\,dt < \infty$ (see Hypothesis 3.8) but instead impose throughout a rather restrictive condition $\int_{\mathbb{R}} \|B(t)\|_2\,dt < 1$ (see Hypothesis 3.2) and leave the general case as an open problem. A detailed discussion of the related issues can be found at the end of Section 3. We have made however a significant step in solving this problem by proving that a respective integral operator (whose null space is given by the Jost solutions) is Fredholm with index zero.

Next, we will outline the main issues addressed in this paper and its structure in more details. We consider equations (1.1) with a unbounded operator A having discrete semi–simple spectrum, and fix the orthonormal basis of its eigenvectors so that A becomes diagonal. Using the same notations $\varkappa_j$, Q_j, $\mathcal{Q}_\pm$, $T_\pm(\cdot)$ as above, we now remark that the sequence $\varkappa_j$, $j \in \mathbb{Z}_\pm$ and the dimensions of $\operatorname{im} \mathcal{Q}_\pm$ are infinite. We impose rather strong assumptions on the growth of $\varkappa_j$ and the gaps between $\varkappa_j$ and $\varkappa_k$, see Hypothesis 2.2 (we believe that these assumptions can be relaxed, but choose not to pursue this here). The operators $B(t)$ in (1.1) are assumed to be Hilbert–Schmidt, $\|B(\cdot)\|_2$ is summable and $B(\cdot)$ is a piecewise strongly continuous operator valued function. In addition, we impose a technical assumption $\int_{\mathbb{R}_\pm} \|B(t)\|_2\,dt < 1$ needed to make sure that the operator valued Jost solutions can be extended from the sets $\{\pm t \geq \tau\}$, with a large τ, to $\mathbb{R}_\pm$.The backward/forward extension issue is of course a totally new infinite dimensional problem typical for the ill–posed equations (1.1).

In Section 2 we introduce and study the Birman–Schwinger type operator, $\mathcal{K}$, whose kernel is given by the formula $K(t,s) = B_r(t)\mathcal{V}(t-s)B_\ell(s)$, $t,s \in \mathbb{R}$. Here $\mathcal{V}$ is the main Green's function of the unperturbed equation in (1.1), so that $u =$

$\mathcal{V} * f$ is the (mild) solution of the inhomogeneous equation $u'(t) = Au(t) + f(t)$, $f \in L^2(\mathbb{R}, \mathcal{X})$, and the operators $B_{r,\ell}(t)$ are chosen to factorize $B(t) = B_\ell(t)B_r(t)$. In addition, we study the "truncated" operator $\mathcal{K}_n$ corresponding to a finitely supported finite dimensional Galerkin approximation $B_n(t)$ of the perturbation $B(t)$ obtained by projecting it on the range of the projection $\mathcal{Q}^{(n)} = \sum_{|j|\leq n} Q_j$. The main result of Section 2 concerns the convergence of $\mathcal{K}_n$ to $\mathcal{K}$ in the Hilbert–Schmidt norm as $n \to \infty$. This result is crucial in approximating the infinite dimensional Evans function by the finite dimensional truncations.

In Section 3 we prove the existence and uniqueness of (operator valued) solutions of certain Fredholm integral equations; these solutions are called the generalized operator valued Jost solutions $Y_\pm^{(j)}(\cdot)$, $j \in \mathbb{Z}_\mp$, on $\mathbb{R}_\pm$ of the perturbed equation (1.1). A nice property of these solutions is the decay $\|Y_\pm^{(j)}(t)\| \leq ce^{\varkappa_j t}$, $t \in \mathbb{R}_\pm$, with a constant $c > 0$ independent of j. Our objective, however, is to construct the operators $\mathcal{Y}_\pm = \sum_{j\in\mathbb{Z}_\mp} Y_\pm^{(j)}(0)$. The convergence of the latter series is a new infinite dimensional challenge taken care of in Theorem 3.6, where we also prove that the operators $\mathcal{Y}_\pm - \mathcal{Q}_\pm$ are Hilbert–Schmidt. This allows us to define the infinite dimensional Evans determinant, $E = \det_{2,\mathcal{X}}(\mathcal{Y}_+ + \mathcal{Y}_-)$, as a 2–modified determinant in $\mathcal{X}$ (versus the usual determinant $E = \det(\mathcal{Y}_+ + \mathcal{Y}_-)$ in the finite dimensional case). Passing to the 2–modified determinant in the definition of the Evans function is one of the major infinite dimensional innovations of the current paper. In addition, in Theorem 3.9 we prove that the integral operator associated with the Fredholm integral equations is Fredholm with index zero.

The main objective of Section 4 is to develop machinery that allows one to approximate in the Hilbert–Schmidt norm the operator $\mathcal{Y}_+ + \mathcal{Y}_-$ by a natural extension to the space $\mathcal{X}$ of the finite dimensional Galerkin truncations $\mathcal{Y}_{+,n} + \mathcal{Y}_{-,n}$ (Lemma 4.3). As a result, we show in Theorem 4.4 that the sequence of the 2–modified Evans functions E_n for the truncated systems converges to the 2–modified Evans function E of the full perturbed equations (1.1).

Section 5 contains results leading to the proof of our main formula (1.4), (1.5), see Theorem 5.8. The idea is to use (1.2) for the truncated system, and then compute the limit as $n \to \infty$. Passing in the RHS of (1.2), written for the truncated system, to the 2–modified determinant, and recalling the general formula $\det_2(I + T) = e^{-\operatorname{tr}(T)} \det(I + T)$, we infer:

$$\det_{2,L^2(\mathbb{R},\mathcal{X})}(I - \mathcal{K}_n) = e^{\Theta^{(n)} - \Theta^{(n)}_{\text{mod}}} \det_2(\mathcal{Y}_{+,n} + \mathcal{Y}_{-,n}), \tag{1.6}$$

where the modification $\Theta^{(n)}_{\text{mod}} = \operatorname{tr}(I - \mathcal{Y}_{+,n} - \mathcal{Y}_{-,n})$ is the trace taken in the range of the finite dimensional projection $\mathcal{Q}^{(n)}$, and $\Theta^{(n)}$ is given by (1.3) with B replaced by B_n and $\mathcal{Q}_\pm$ replaced by $\mathcal{Q}^{(n)}\mathcal{Q}_\pm$. From Section 2 we know that the LHS of (1.6) converges to $\det_{2,L^2(\mathbb{R},\mathcal{X})}(I - \mathcal{K})$. From Section 4 we know that $\det_2(\mathcal{Y}_{+,n} + \mathcal{Y}_{-,n})$ in the RHS of (1.6) converge to $\det_{2,\mathcal{X}}(\mathcal{Y}_+ + \mathcal{Y}_-)$. The proof of convergence of $\Theta^{(n)} - \Theta^{(n)}_{\text{mod}}$ to Ξ defined in (1.5) requires some careful computation of the trace–class norms, and culminates in Theorem 5.5.

In Section 6 we apply our results to equations (1.1) rooted in eigenvalue problems for a class of second order differential operators, $\mathcal{L}$, with operator coefficients, of the form $\mathcal{L} = \partial_t^2 + (B_1(t) - c)\partial_t + (A_0 + B_0(t))$ (in fact, we deal with a substantially more general class of operators, see (6.2)). Here, A_0 is an unbounded, self–adjoint, negative operator with discrete spectrum and B_0, B_1 are bounded, strongly continuous, operator valued functions with Hilbert–Schmidt values. Re–writing the eigenvalue problem $\mathcal{L}U = \lambda U$ as the perturbed equation in (1.1), we recalculate the eigenvalues of $A = A(\lambda)$ via the eigenvalues of A_0 and find an open set $\Omega \subset \mathbb{C}$ for which $\mathcal{V}(\lambda)$ and $\mathcal{K}(\lambda)$ are holomorphic in $\lambda \in \Omega$. Under appropriate conditions, it turns out that the truncated projections $\mathcal{Q}^{(n)} = \mathcal{Q}^{(n)}(\lambda)$ are holomorphic for $\lambda \in \Omega$ for n sufficiently large. The key argument of the proof is the separation of the eigenvalues $a_n(\lambda)$ of $A(\lambda)$ with n large enough, which is a consequence of Hypothesis 2.2. The latter fact allows one to show that $\Xi(\lambda)$ is holomorphic, although the proof of this relies on the approximation techniques developed in Section 5. Now formula (1.4) yields that $E(\lambda)$ is a holomorphic function on Ω. This, and the fact that $E(\lambda) = 0$ if and only if λ is an eigenvalue of $\mathcal{L}$ is proved in Theorem 6.13

Section 7 contains two examples (a 6–th order two dimensional PDE and a 5-th order Kadomtsev-Petviashvili equation) aimed to illustrate how to check the hypothesis needed to guarantee the analyticity of the Evans function and its connection to the Fredholm determinants.

A glossary of notation: For natural numbers d, d_1, d_2, let $\mathbb{C}^{d_1\times d_2}$ be the set of $d_1 \times d_2$ matrices with complex entries; by $\operatorname{tr}_{\mathbb{C}^d}(M)$ and $\det_{\mathbb{C}^d}(M)$ we denote the trace and determinant of a $d \times d$ matrix $M \in \mathbb{C}^{d\times d}$. $\|\cdot\|_{\mathbb{C}^d}$ denotes a vector norm in $\mathbb{C}^d$; $\|\cdot\|_{\mathbb{C}^{d\times d}}$ a matrix norm in $\mathbb{C}^{d\times d}$; $\|\cdot\|_{\mathcal{X}}$ a norm in a Banach space $\mathcal{X}$. For $p \geq 1$ and $J \subseteq \mathbb{R}$, $L^p(J, \mathcal{X})$ are the usual Lebesgue spaces on J with values in $\mathcal{X}$, associated with Lebesgue measure dx on J. Similarly, $L^p(J, \mathcal{X}; w(x)dx)$ are the weighted spaces with a weight $w \geq 0$. The respective spaces of bounded continuous functions on J are denoted by $C_{\mathrm{b}}(J, \mathcal{X})$ and $C_{\mathrm{b}}(J, \mathcal{X}; w(x))$. $H^s(\mathbb{R}, \mathcal{X})$, $s > 0$, is the usual Sobolev space of $\mathcal{X}$ valued functions. The identity matrix in $\mathbb{C}^{d\times d}$ is denoted by I_d and the identity operator on a Banach space $\mathcal{X}$ is denoted by I (or by $I_{\mathcal{X}}$ if its dependence on $\mathcal{X}$ needs to be stressed). The set of bounded linear operators from a Hilbert space $\mathcal{X}$ to itself is denoted by $\mathcal{B}(\mathcal{X})$. For an operator T on a Hilbert space $\mathcal{X}$ we use T^*, $\operatorname{dom}(T)$, $\ker T$, $\operatorname{im} T$, $\sigma(T)$, $\rho(T)$, $\sigma_F(T)$, $R(\lambda, T) = (\lambda - T)^{-1}$ and $T_{|\mathcal{Y}}$ to denote the adjoint, domain, kernel, range, spectrum, resolvent set, Fredholm spectrum, resolvent operator and the restriction of T on a subspace $\mathcal{Y}$ of $\mathcal{X}$. If T is a linear operator on $\mathcal{X}$, then $T^{-1} \in \mathcal{B}(\mathcal{X})$ denotes the (bounded) inverse operator of T. If $B : J \to \mathcal{B}(\mathcal{X})$ then M_B denotes the operator of multiplication by $B(\cdot)$ in $L^p(J, \mathcal{X})$ or $C_{\mathrm{b}}(J, \mathcal{X})$. If $\mathcal{X}_1$ and $\mathcal{X}_2$ are two subspaces of $\mathcal{X}$, then $\mathcal{X}_1 \oplus \mathcal{X}_2$ denotes their direct (but not necessarily orthogonal) sum. $\mathcal{B}_p(\mathcal{X})$ denotes the Schatten–von Neumann ideals of compact operators on a Hilbert space $\mathcal{X}$ with singular values in ℓ^p, $p \geq 1$. We abbreviate $\|\cdot\|$ for the norm in $\mathcal{B}(\mathcal{X})$ and $\|\cdot\|_p$ for the norm in $\mathcal{B}_p(\mathcal{X})$. $\mathcal{B}_\infty(\mathcal{X})$ denotes the space of compact operators on $\mathcal{X}$. We denote by $\mathcal{C}_p^\pm = C_{\mathrm{b}}(\mathbb{R}_\pm; \mathcal{B}_p(\mathcal{X}))$, $1 \leq p \leq \infty$ and $\mathcal{C}_{\mathrm{b}}^\pm = C_{\mathrm{b}}(\mathbb{R}_\pm; \mathcal{B}(\mathcal{X}))$ the space of continuous and bounded functions on $\mathbb{R}_\pm$ with values in $\mathcal{B}_p(\mathcal{X})$ and $\mathcal{B}(\mathcal{X})$. We write $\det_{k,\mathcal{X}}(T)$, $k \in \mathbb{Z}_+ \setminus \{0\}$, and $\operatorname{tr}_{\mathcal{X}}(T)$ for the (k–modified) Fredholm determinant and trace of the operator T in a Hilbert space $\mathcal{X}$, and abbreviate $\det_k(T)$ and $\operatorname{tr}(T)$ when possible. A generic constant is denoted by c or C, and we also denote $\mathbb{Z}_+ = \{1, 2, \dots\}$, $\mathbb{Z}_- = \{-1, -2, \dots\}$, and $\mathbb{R}_+ = [0, \infty)$, $\mathbb{R}_- = (-\infty, 0]$. The ε-disc in $\mathbb{C}$ centered at λ is denoted by $B(\lambda, \varepsilon)$. The Fourier transform $\mathcal{F}$ is defined by $(\mathcal{F}\mu)(\xi) = \int_{\mathbb{R}} e^{-2\pi i\xi t} d\mu(t)$ for Borel measure μ.

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2. Setting and Preliminary results

Let $\mathcal{X}_+$ and $\mathcal{X}_-$ be separable Hilbert spaces, and let $\{v_n : n \in \mathbb{Z}_\pm\}$ be an orthonormal Hilbert basis in $\mathcal{X}_\pm$ and let $(a_n)_{n\in\mathbb{Z}}$ be a sequence of complex numbers such that:

$$\inf_{n\in\mathbb{Z}} \operatorname{Re} a_n > 0; \tag{2.1}$$

$$\sum_{n\in\mathbb{Z}} (\operatorname{Re} a_n)^{-\gamma} < \infty \quad \text{for some} \quad \gamma > 0. \tag{2.2}$$

On the space $\mathcal{X}_+$, respectively, $\mathcal{X}_-$, we define a strongly continuous semigroup, $\{T_+(t)\}_{t\geq 0}$, respectively, $\{T_-(t)\}_{t\geq 0}$, of bounded linear operators by

$$T_+(t)v_n = e^{-a_n t}v_n,\ n \geq 1, \quad \text{respectively} \quad T_-(t)v_n = e^{-a_n t}v_n,\ n \leq -1,\ t \geq 0. \tag{2.3}$$

The generator A_+, respectively, A_-, of this semigroup is then the operator of multiplication by $(-a_n)_{n\in\mathbb{Z}_+}$, respectively, $(-a_n)_{n\in\mathbb{Z}_-}$; that is,

$$A_+v_n = -a_n v_n, \quad n \geq 1 \quad \text{and} \quad A_-v_n = -a_n v_n, \quad n \leq -1. \tag{2.4}$$

Clearly, if (2.1) holds then

$$\|T_+(t)\| = e^{-(\inf_{n\geq 1} \operatorname{Re} a_n)t}, \quad \|T_-(t)\| = e^{-(\inf_{n\leq -1} \operatorname{Re} a_n)t}, t \geq 0. \tag{2.5}$$

If (2.2) holds then $e^{-a_n t} \to 0$ as $n \to \pm\infty$ for each $t > 0$, and thus the operators $T_\pm(t)$ are compact for each $t > 0$. By a standard semigroup result, see, e. g. [13, Thm. II.4.29], the compactness is equivalent to the facts that the resolvents of A_+ and A_- are compact operators, and the semigroups $\{T_\pm(t)\}_{t\geq 0}$ are norm continuous for $t > 0$. The following simple calculation shows that, moreover, $T_\pm(t)$ belong to the Hilbert-Schmidt class.

Lemma 2.1. *If* (2.1) *and* (2.2) *hold then* $T_\pm(t) \in \mathcal{B}_2(\mathcal{X}_\pm)$ *for every* $t > 0$*, and, moreover, for* $t > 0$ *we have the estimates:*

$$\|T_+(t)\|_2 \leq c\alpha_0 t^{-\gamma/2}, \quad \textit{where} \quad \alpha_0 = \Big(\sum_{n=1}^{\infty} (\operatorname{Re} a_n)^{-\gamma}\Big)^{1/2}, \tag{2.6}$$

$$\|T_-(t)\|_2 \leq c\beta_0 t^{-\gamma/2}, \quad \textit{where} \quad \beta_0 = \Big(\sum_{n=-\infty}^{-1} (\operatorname{Re} a_n)^{-\gamma}\Big)^{1/2}. \tag{2.7}$$

Proof. If $a, \gamma > 0$ then

$$\max_{t\geq 0}(t^\gamma e^{-at}) = a^{-\gamma}(\gamma/e)^\gamma. \tag{2.8}$$

Using (2.2), for the orthonormal basis $\{v_n : n \in \mathbb{Z}_+\}$ we have:

$$\sum_{n=1}^{\infty} \|T_+(t)v_n\|^2_{\mathcal{X}_+} = \sum_{n=1}^{\infty} e^{-2\operatorname{Re} a_n t} \leq (\gamma/e)^\gamma t^{-\gamma} \sum_{n=1}^{\infty} (2\operatorname{Re} a_n)^{-\gamma} < \infty, \quad t > 0. \tag{2.9}$$

By the standard properties of the Hilbert-Schmidt operators, cf. [37, Sec. 1.6.5], it follows that $T_+(t) \in \mathcal{B}_2(\mathcal{X}_+)$ and that the required estimate for $\|T_+(t)\|_2$ holds. The argument for $T_-(t)$ is similar. □

Next, we define $\mathcal{X} = \mathcal{X}_+ \oplus \mathcal{X}_-$ and consider on $\mathcal{X}$ the operator $A = A_+ \oplus (-A_-)$, the direct sum of the operators defined in (2.4). The operator A is called the generator of the bi-semigroup $\{T_\pm(t)\}_{t\geq 0}$, cf. [2]. We denote by $\mathcal{Q}_\pm$ the projections of $\mathcal{X}$ onto $\mathcal{X}_\pm$, associated with the direct sum decomposition $\mathcal{X} = \mathcal{X}_+ \oplus \mathcal{X}_-$ so that $\mathcal{X}_+ = \operatorname{im}\mathcal{Q}_+$ and $\mathcal{X}_- = \operatorname{im}\mathcal{Q}_-$. The choice of plus and minus signs in these notations is consistent with the fact that $\|T_+(t)\| \to 0$ as $t \to +\infty$ (on $\mathbb{R}_+$) while $\|T_-(-t)\| \to 0$ as $t \to -\infty$ (on $\mathbb{R}_-$). We define $\mathcal{V} : \mathbb{R} \to \mathcal{B}(\mathcal{X})$ by

$$\mathcal{V}(t) = T_+(t)\mathcal{Q}_+ \quad \text{for} \quad t \geq 0 \quad \text{and} \quad \mathcal{V}(t) = -T_-(-t)\mathcal{Q}_- \quad \text{for} \quad t < 0, \tag{2.10}$$

and note that (cf. (2.1))

$$\|\mathcal{V}(t)\| \leq e^{-c|t|}, \quad \text{for all} \quad t \in \mathbb{R} \quad \text{for some} \quad c > 0. \tag{2.11}$$

The Cauchy problem $u'(t) = Au(t)$, $u(0) = u_0$, in $\mathcal{X}$ has a (forward) solution for $t \geq 0$ if and only if $u_0 \in \mathcal{X}_+$ and a (backward) solution for $t \leq 0$ if and only if $u_0 \in \mathcal{X}_-$. If $f \in L^2(\mathbb{R}, \mathcal{X})$ then, passing to the Fourier transform, the solution of the inhomogeneous equation $u'(t) = Au + f$ on $\mathbb{R}$ is given as the convolution $u = \mathcal{V} * f$.

We will now introduce spectral projections for the semigroups $\{T_\pm(t)\}_{t\geq 0}$. Assuming (2.1)-(2.2), we enumerate the elements of the sequence $(a_n)_{n\in\mathbb{Z}}$ such that $\operatorname{Re} a_1 \leq \operatorname{Re} a_2 \leq \dots$ and $\operatorname{Re} a_{-1} \leq \operatorname{Re} a_{-2} \leq \dots$. We will cluster together the complex numbers a_n's having equal values of their real parts, and will denote these values by $\varkappa_j$, when $j \in \mathbb{Z}_+$, and by $-\varkappa_j$, when $j \in \mathbb{Z}_-$. Thus, the sequence $(\varkappa_j)_{j\in\mathbb{Z}\setminus\{0\}}$ is ordered as follows:

$$\dots < \varkappa_{-2} < \varkappa_{-1} < 0 < \varkappa_1 < \varkappa_2 < \dots. \tag{2.12}$$

Next, we define the following sets of integers:

$$N_j = \{n \in \mathbb{Z}_- : \operatorname{Re} a_n = -\varkappa_j\},\ j \leq -1; \quad N_j = \{n \in \mathbb{Z}_+ : \operatorname{Re} a_n = \varkappa_j\},\ j \geq 1. \tag{2.13}$$

For $j \in \mathbb{Z}\setminus\{0\}$ we denote by Q_j the orthogonal projection in $\mathcal{X}_+$, respectively, $\mathcal{X}_-$ onto the span of the basis vectors v_n, with $n \in N_j$. We remark that N_j is finite for each $j \in \mathbb{Z}\setminus\{0\}$ by (2.2) and thus each Q_j has finite rank:

$$\dim(\operatorname{im} Q_j) < \infty \quad \text{for each} \quad j \in \mathbb{Z}\setminus\{0\}. \tag{2.14}$$

This numeration of $\varkappa_j$'s and the spectral projections Q_j's for the operator $A = A_+ \oplus (-A_-)$ is consistent since Q_j with the negative (respectively, positive) number $j \in \mathbb{Z}\setminus\{0\}$ corresponds to the spectral subset of A consisting of complex numbers having negative (respectively, positive) real parts, so that

$$\|T_+(t)Q_j\| = e^{\varkappa_j t} \to 0 \ \text{ as } \ t \to +\infty\,, \quad j \in \mathbb{Z}_-, \tag{2.15}$$

$$\|T_-(-t)Q_j\| = e^{\varkappa_j t} \to 0 \ \text{ as } \ t \to -\infty\,, \quad j \in \mathbb{Z}_+. \tag{2.16}$$

Throughout this paper we assume the following conditions on $\varkappa_j$, $j \in \mathbb{Z}\setminus\{0\}$:

Hypothesis 2.2. *In addition to* (2.1)–(2.2) *we assume*

$$\sum_{(k,j)\in\mathbb{Z}_{2,\pm}} (\varkappa_j - \varkappa_k)^{-2} < \infty,$$

where we introduce the following notations:

$$\begin{aligned} \mathbb{Z}_{2,+} &= \{(k,j) \in \mathbb{Z}^2 : j \leq -1,\ k > j,\ k \neq 0\}, \\ \mathbb{Z}_{2,-} &= \{(k,j) \in \mathbb{Z}^2 : j \geq 1,\ k < j,\ k \neq 0\}. \end{aligned} \tag{2.17}$$

Let $B:\mathbb{R}\to\mathcal{B}(\mathcal{X})$ be a strongly continuous operator valued function. Let $B(t)=U(t)|B(t)|$ be the polar decomposition of $B(t)$, $t\in\mathbb{R}$, and let $B_\ell, B_r:\mathbb{R}\to\mathcal{B}(\mathcal{X})$ be defined by

$$B_\ell(t)=U(t)|B(t)|^{1/2},\quad B_r(t)=|B(t)|^{1/2} \tag{2.18}$$

so that $B(t)=B_\ell(t)B_r(t)$ for all $t\in\mathbb{R}$. Let

$$\mathcal{K}=M_{B_r}(\mathcal{V}*M_{B_\ell}) \tag{2.19}$$

be the integral operator on $L^2(\mathbb{R},\mathcal{X})$ with the kernel $K:\mathbb{R}^2\to\mathcal{B}(\mathcal{X})$ given by

$$K(t,s)=B_r(t)\mathcal{V}(t-s)B_\ell(s),\quad t,s\in\mathbb{R}. \tag{2.20}$$

The operator $\mathcal{K}$ is called the Birman–Schwinger operator, cf. [14, 16, 19]. Using Lemma 2.1, it follows that $K(t,s)\in\mathcal{B}_2(\mathcal{X})$ and

$$\|K(t,s)\|_2\le c\|B(t)\|^{1/2}\|B(s)\|^{1/2}|t-s|^{-\gamma/2}\quad\text{for all}\quad t,s\in\mathbb{R},\, t\ne s. \tag{2.21}$$

In the sequel we will frequently use a well-know result, see [37, Sec. 1.6.5] and cf. [31, Thm. VI.23], saying that the operator $\mathcal{K}$ is Hilbert-Schmidt on $L^2(\mathbb{R},\mathcal{X})$ provided the operators $K(t,s)$ are Hilbert-Schmidt on $\mathcal{X}$ and $\|K(\cdot\,,\cdot)\|_2$ is square-summable on $\mathbb{R}^2$; moreover,

$$\|\mathcal{K}\|^2_{\mathcal{B}_2(L^2(\mathbb{R},\mathcal{X}))}=\int_{\mathbb{R}^2}\|K(t,s)\|^2_{\mathcal{B}_2(\mathcal{X})}\,dt\,ds. \tag{2.22}$$

Also, we will use the Hardy-Littlewood inequality, see, e.g., [32, p. 30]: If $p,q>1$, $\lambda\in(0,1)$, $p^{-1}+q^{-1}+\lambda=2$, $f\in L^p(\mathbb{R})$ and $g\in L^q(\mathbb{R})$ then

$$\int_{\mathbb{R}^2}|f(t)g(\tau)||t-\tau|^{-\lambda}\,dt\,d\tau\le C_{p,q,\lambda}\|f\|_p\|g\|_q; \tag{2.23}$$

and the Riesz composition formula, see e. g. [12, Thm. 3.1]: If $d\in\mathbb{Z}_+$, $d/2<\alpha<d$ then there exists $c>0$ such that

$$\int_{\mathbb{R}^d}\|z-x\|^{-\alpha}_{\mathbb{C}^d}\|z-y\|^{-\alpha}_{\mathbb{C}^d}\,dz=c\|x-y\|^{d-2\alpha}_{\mathbb{C}^d}. \tag{2.24}$$

In the next lemma we give several conditions that guarantee that $\mathcal{K}$ belongs to the ideals $\mathcal{B}_2(L^2(\mathbb{R},\mathcal{X}))$ or $\mathcal{B}_4(L^2(\mathbb{R},\mathcal{X}))$. Assertion (iii) will be frequently used below.

Lemma 2.3. *Assume conditions* (2.1)–(2.2).

(i) If $\gamma\in(0,1)$ and $\|B(\cdot)\|\in L^p(\mathbb{R})$ with $p=2/(2-\gamma)$ then $\mathcal{K}\in\mathcal{B}_2(L^2(\mathbb{R},\mathcal{X}))$;

(ii) If $\gamma\in(1,3/2)$ $\|B(\cdot)\|\in L^p(\mathbb{R})\cap L^\infty(\mathbb{R})$ with $p=1/(2-\gamma)$ then $\mathcal{K}\in\mathcal{B}_4(L^2(\mathbb{R},\mathcal{X}))$;

(iii) If $B(t)\in\mathcal{B}_2(\mathcal{X})$ for all $t\in\mathbb{R}$ and $\|B(\cdot)\|_2\in L^1(\mathbb{R})$ then $\mathcal{K}\in\mathcal{B}_2(L^2(\mathbb{R},\mathcal{X}))$;

(iv) If $B(t)\in\mathcal{B}_4(\mathcal{X})$ for all $t\in\mathbb{R}$ and $\|B(\cdot)\|_4\in L^1(\mathbb{R})$ then $\mathcal{K}\in\mathcal{B}_4(L^2(\mathbb{R},\mathcal{X}))$.

Proof. (i) This assertion follows from (2.22) since $\|K(\cdot\,,\cdot)\|_2$ is square-summable by (2.21) and (2.23).

(ii) Starting the proof, we note that the adjoint to $\mathcal{K}$ is the integral operator $(\mathcal{K}^*f)(t)=\int_{\mathbb{R}}K(s,t)^*f(s)\,ds$. Thus, the kernel of the operator $\mathcal{K}^*\mathcal{K}$ is given by $L(t,\tau)x=\int_{\mathbb{R}}K(s,t)^*K(s,\tau)x\,ds$. By (2.21), for all $t\ne s\ne\tau\ne t$ we have:

$$\begin{aligned}\|K(s,t)^*K(s,\tau)\|_2&\le c\|K(s,t)^*\|_2\|K(s,\tau)\|_2=c\|K(s,t)\|_2\|K(s,\tau)\|_2\\&\le c\|B(t)\|^{1/2}\|B(\tau)\|^{1/2}\|B(s)\|\,|t-s|^{-\gamma/2}|s-\tau|^{-\gamma/2}\\&\le c\|B(t)\|^{1/2}\|B(\tau)\|^{1/2}|t-s|^{-\gamma/2}|s-\tau|^{-\gamma/2}.\end{aligned}$$

Taking $d = 1$ and $\alpha = \gamma/2 \in (1/2, 1)$ in (2.24), we infer:

$$\int_{\mathbb{R}} \|K(s,t)^*K(s,\tau)\|_2 ds \leq c\|B(t)\|^{1/2}\|B(\tau)\|^{1/2}|t-\tau|^{1-\gamma} < \infty \quad \text{for all} \quad t \neq \tau.$$

By Lemma A.1 we have $L(t,\tau) \in \mathcal{B}_2(\mathcal{X})$ and $\|L(t,\tau)\|_2 \leq c\|B(t)\|^{1/2}\|B(\tau)\|^{1/2}|t-\tau|^{1-\gamma}$ for $t \neq \tau$. Since $\|B(\cdot)\| \in L^p(\mathbb{R})$, with $p = \frac{1}{2-\gamma}$ and $\lambda = 2(\gamma-1) \in (0,1)$, from the Hardy-Littlewood inequality (2.23) we conclude:

$$\int_{\mathbb{R}^2} \|L(t,\tau)\|_2^2 \, dt d\tau \leq c \int_{\mathbb{R}^2} \|B(t)\| \, \|B(\tau)\| |t-\tau|^{-\lambda} dt d\tau < \infty.$$

Thus, $\mathcal{K}^*\mathcal{K} \in \mathcal{B}_2(L^2(\mathbb{R},\mathcal{X}))$ by (2.22), proving (ii).

(iii) Since $B(t) \in \mathcal{B}_2(\mathcal{X})$, for the operators defined in (2.18) we infer that $B_\ell(t), B_r(t) \in \mathcal{B}_4(\mathcal{X})$ and $\|B_\ell(t)\|_4 = \|B_r(t)\|_4 = \|B(t)\|_2^{1/2}$. Using (2.22) and estimate (2.11),

$$\begin{aligned}\|K(t,s)\|_2 &\leq \|B_r(t)\|_4\|\mathcal{V}(t-s)B_\ell(s)\|_4 \leq \|B_r(t)\|_4\|\mathcal{V}(t-s)\|\|B_\ell(s)\|_4 \\ &\leq \|B(t)\|_2^{1/2}e^{-c|t-s|}\|B(s)\|_2^{1/2} \leq \|B(t)\|_2^{1/2}\|B(s)\|_2^{1/2},\end{aligned}$$

and assertion (iii) follows.

(iv) Since $B(t) \in \mathcal{B}_4(\mathcal{X})$, we conclude that $B_\ell(t), B)r(t) \in \mathcal{B}_8(\mathcal{X})$ and $\|B_\ell(t)\|_8 = \|B_r(t)\|_8 = \|B(t)\|_4^{1/2}$. As before,

$$\begin{aligned}\|K(t,s)\|_4 &\leq \|B_r(t)\|_8\|\mathcal{V}(t-s)B_\ell(s)\|_8 \leq \|B_r(t)\|_8\|\mathcal{V}(t-s)\|\|B_\ell(s)\|_8 \\ &\leq \|B(t)\|_4^{1/2}e^{-c|t-s|}\|B(s)\|_4^{1/2} \leq \|B(t)\|_4^{1/2}\|B(s)\|_4^{1/2}\end{aligned}$$

for all $t, s \in \mathbb{R}$, which implies

$$\begin{aligned}\|K(s,t)^*K(s,\tau)\|_2 &\leq \|K(s,t)^*\|_4\|K(s,\tau)\|_4 = c\|K(s,t)\|_4\|K(s,\tau)\|_4 \\ &\leq \|B(t)\|_4^{1/2}\|B(\tau)\|_4^{1/2}\|B(s)\|_4\end{aligned}$$

for all $t, s, \tau \in \mathbb{R}$. We infer

$$\begin{aligned}\int_{\mathbb{R}} \|K(s,t)^*K(s,\tau)\|_2 \, ds &\leq \|B(t)\|_4^{1/2}\|B(\tau)\|_4^{1/2} \int_{\mathbb{R}} \|B(s)\|_4 \, ds \\ &= c\|B(t)\|_4^{1/2}\|B(\tau)\|_4^{1/2}\end{aligned}$$

for all $t, \tau \in \mathbb{R}$. By Lemma A.1 we have $L(t,\tau) \in \mathcal{B}_2(\mathcal{X})$ and $\|L(t,\tau)\|_2 \leq c\|B(t)\|_4^{1/2}\|B(\tau)\|_4^{1/2}$ for all $t, \tau \in \mathbb{R}$. Moreover,

$$\int_{\mathbb{R}^2} \|L(t,\tau)\|_2^2 \, dt \, d\tau \leq c \int_{\mathbb{R}^2} \|B(t)\|_4\|B(\tau)\|_4 \, dt \, d\tau = c\Big(\int_{\mathbb{R}} \|B(t)\|_4 \, dt\Big)^2.$$

From (2.22) we infer $\mathcal{K}^*\mathcal{K} \in \mathcal{B}_2(L^2(\mathbb{R},\mathcal{X}))$, proving $\mathcal{K} \in \mathcal{B}_4(L^2(\mathbb{R},\mathcal{X}))$. □

In Section 3 we will extend to the infinite dimensional case the results from [14] on the computation of the (modified) Fredholm determinant of the operator $I - \mathcal{K}$ via solutions of certain integral equations. For this we will need the following approximation of the integral operator $\mathcal{K}$ by integral operators $\mathcal{K}_n$ having semi-separable integral kernels for which the theory developed in [14, 18] is readily available. We introduce in $\mathcal{X}_+$, $\mathcal{X}_-$, and $\mathcal{X}$ the orthogonal projections

$$\mathcal{Q}_+^{(n)} = \sum_{k=-n}^{-1} Q_k, \; \mathcal{Q}_-^{(n)} = \sum_{k=1}^{n} Q_k, \; \text{ and } \; \mathcal{Q}^{(n)} = \mathcal{Q}_+^{(n)} \oplus \mathcal{Q}_-^{(n)}, \, n \in \mathbb{Z}_+ \setminus \{0\}, \quad (2.25)$$

where Q_j's are the spectral projections for A satisfying (2.14). In the sequel we will use the notation

$$d(n) = \dim(\operatorname{im} \mathcal{Q}^{(n)}). \tag{2.26}$$

We note that $\mathcal{Q}_\pm^{(n)} = \mathcal{Q}_\pm^{(n)}\mathcal{Q}_\pm = \mathcal{Q}_\pm\mathcal{Q}_\pm^{(n)}$ as projections in $\mathcal{X}$. Since strongly continuous semigroups on finite dimensional spaces can be extended naturally to strongly continuous groups, we now have strongly continuous families $\{T_\pm(t)\mathcal{Q}_\pm^{(n)}\}_{t\in\mathbb{R}}$ of operators on $\mathcal{X}_\pm$, that can be defined for all $t \in \mathbb{R}$ by the same formulas (2.3). Obviously, the operators $T_\pm(t)\mathcal{Q}_\pm^{(n)}$, $t \in \mathbb{R}$, and $A_\pm\mathcal{Q}_\pm^{(n)}$ are operators of finite rank. Since $T_\pm(t)$, $t \geq 0$, and $\mathcal{Q}_\pm^{(n)}$ commute, we conclude, see (2.10), that $\mathcal{V}(t-s)$ and $\mathcal{Q}^{(n)}$, $t, s \in \mathbb{R}$, also commute. Therefore, for $t, s \in \mathbb{R}, n \in \mathbb{Z}_+$ we have:

$$\mathcal{Q}^{(n)}\mathcal{V}(t-s)\mathcal{Q}^{(n)} = \begin{cases} T_+(t)\mathcal{Q}_+^{(n)} \cdot T_+(-s)\mathcal{Q}_+^{(n)}, & \text{if} \quad t \geq s, \\ -T_-(-t)\mathcal{Q}_-^{(n)} \cdot T_-(s)\mathcal{Q}_-^{(n)}, & \text{if} \quad t < s. \end{cases} \tag{2.27}$$

Further, let φ_n be a C^∞ function with compact support such that $0 \leq \varphi_n \leq 1$, $\varphi_n(t) = 1$ for all $t \in [-n, n]$ and $\varphi_n(t) = 0$ for all $t \notin [-n-1, n+1]$. Let $\mathcal{K}_n$ be the integral operator on $L^2(\mathbb{R}, \mathcal{X})$ with the integral kernel defined by

$$K_n(t,s) = \varphi_n(t)\varphi_n(s)B_r(t)\mathcal{Q}^{(n)}\mathcal{V}(t-s)\mathcal{Q}^{(n)}B_\ell(s), \quad t, s \in \mathbb{R}. \tag{2.28}$$

We refer to [18] and the literature therein for the theory of integral operators with semi–separable kernels. Specifically, we introduce the notations

$$\begin{aligned} f_{1,n}(t) &:= \varphi_n(t)B_r(t)T_+(t)\mathcal{Q}_+^{(n)}\mathcal{Q}_+ : \mathcal{X}_+ \to \mathcal{X}, \\ f_{2,n}(t) &:= \varphi_n(t)B_r(t)T_-(-t)\mathcal{Q}_-^{(n)}\mathcal{Q}_- : \mathcal{X}_- \to \mathcal{X}, \\ g_{1,n}(s) &:= \varphi_n(s)\mathcal{Q}_+T_+(-s)\mathcal{Q}_+^{(n)}B_\ell(s) : \mathcal{X} \to \mathcal{X}_+, \\ g_{2,n}(s) &:= -\varphi_n(s)\mathcal{Q}_-T_-(s)\mathcal{Q}_-^{(n)}B_\ell(s) : \mathcal{X} \to \mathcal{X}_-. \end{aligned} \tag{2.29}$$

The kernel of $\mathcal{K}_n$ is semi–separable since using (2.27)

$$K_n(t,s) = \begin{cases} f_{1,n}(t)g_{1,n}(s), & t \geq s, \\ f_{2,n}(t)g_{2,n}(s), & t < s. \end{cases} \tag{2.30}$$

Since $\mathcal{Q}^{(n)}$ is of finite rank, the values of the operator-valued functions $f_{j,n}$ and $g_{j,n}$ are operators of finite rank. We remark that $f_{j,n}$ and $g_{j,n}$ are strongly continuous and have compact supports; in particular, they belong to $L^2(\mathbb{R}, \mathcal{B}(\mathcal{X}))$, which is a standing assumption in [18]. In addition, $\mathcal{K}_n \in \mathcal{B}_2(L^2(\mathbb{R}, \mathcal{X}))$ for each n since $\|K_n(\,\cdot, \cdot)\|_2$ is square-summable by (2.5) and the estimate

$$\|K_n(t,s)\|_2 \leq \varphi_n(t)\varphi_n(s)\|B_r(t)\|\,\|\mathcal{V}(t-s)\|\,\|\mathcal{Q}^{(n)}\|_2\,\|B_\ell(s)\| \leq \varphi_n(t)\varphi_n(s)\|\mathcal{Q}^{(n)}\|_2.$$

In the next lemma we give several conditions that guarantee the convergence of $\mathcal{K}_n$ to $\mathcal{K}$ in appropriate norms; assertion (iii) will be frequently used below.

Lemma 2.4. *Assume conditions* (2.1)–(2.2).
(i) If $\gamma \in (0,1)$ *and* $\|B(\cdot)\| \in L^p(\mathbb{R})$ *with* $p = 2/(2-\gamma)$ *then* $\mathcal{K}_n \to \mathcal{K}$ *in* $\mathcal{B}_2(L^2(\mathbb{R}, \mathcal{X}))$ *as* $n \to \infty$*;*
(ii) If $\gamma \in (1, 3/2)$ *and* $\|B(\cdot)\| \in L^p(\mathbb{R}) \cap L^\infty(\mathbb{R})$ *with* $p = 1/(2-\gamma)$ *then* $\mathcal{K}_n \to \mathcal{K}$ *in* $\mathcal{B}_4(L^2(\mathbb{R}, \mathcal{X}))$ *as* $n \to \infty$*;*
(iii) If $B(t) \in \mathcal{B}_2(\mathcal{X})$ *for all* $t \in \mathbb{R}$ *and* $\|B(\cdot)\|_2 \in L^1(\mathbb{R})$ *then* $\mathcal{K}_n \to \mathcal{K}$ *in* $\mathcal{B}_2(L^2(\mathbb{R}, \mathcal{X}))$ *as* $n \to \infty$*;*

(iv) If $B(t) \in \mathcal{B}_4(\mathcal{X})$ for all $t \in \mathbb{R}$ and $\|B(\cdot)\|_4 \in L^1(\mathbb{R})$ then $\mathcal{K}_n \to \mathcal{K}$ in $\mathcal{B}_4(L^2(\mathbb{R},\mathcal{X}))$ as $n \to \infty$.

Proof. Define $\mathcal{V}_n : \mathbb{R} \to \mathcal{B}(\mathcal{X})$ by $\mathcal{V}_n(t) = (I_{\mathcal{X}} - \mathcal{Q}^{(n)})\mathcal{V}(t)(I_{\mathcal{X}} - \mathcal{Q}^{(n)})$. Using (2.6)-(2.7), denoting $\gamma_n^2 = \sum_{|k|\geq n+1}(\operatorname{Re} a_k)^{-\gamma}$, we have $\mathcal{V}_n(t) \in \mathcal{B}_2(\mathcal{X})$ with

$$\|\mathcal{V}_n(t)\|_2^2 \leq ct^{-\gamma}\delta_n^2 \quad t > 0. \tag{2.31}$$

Since

$$\begin{aligned} K_n(t,s) - K(t,s) &= \varphi_n(t)\varphi_n(s)B_r(t)\mathcal{Q}^{(n)}\mathcal{V}(t-s)\mathcal{Q}^{(n)}B_\ell(s) - B_r(t)\mathcal{V}(t-s)B_\ell(s) \\ &= (\varphi_n(t)\varphi_n(s) - 1)B_r(t)\mathcal{Q}^{(n)}\mathcal{V}(t-s)\mathcal{Q}^{(n)}B_\ell(s) - B_r(t)\mathcal{V}_n(t-s)B_\ell(s), \end{aligned} \tag{2.32}$$

using Lemma 2.1 and (2.31) we obtain:

$$\begin{aligned} \|K_n(t,s) - K(t,s)\|_2 &\leq (1 - \varphi_n(t)\varphi_n(s))\|B_r(t)\mathcal{Q}^{(n)}\mathcal{V}(t-s)\mathcal{Q}^{(n)}B_\ell(s)\|_2 \\ &\qquad + \|B_r(t)\mathcal{V}_n(t-s)B_\ell(s)\|_2 \\ &\leq (1 - \varphi_n(t)\varphi_n(s))\|B(t)\|^{1/2}\|\mathcal{V}(t-s)\|_2\|B(s)\|^{1/2} \\ &\qquad + \|B(t)\|^{1/2}\|\mathcal{V}_n(t-s)\|_2\|B(s)\|^{1/2} \\ &\leq c(1 - \varphi_n(t)\varphi_n(s) + \delta_n)\|B(t)\|^{1/2}\,\|B(s)\|^{1/2}|t-s|^{-\gamma/2}. \end{aligned} \tag{2.33}$$

Proof of (i). Using (2.22) and (2.33), we compute

$$\begin{aligned} \|\mathcal{K}_n - \mathcal{K}\|_2^2 &= \int_{\mathbb{R}^2} \|K_n(t,s) - K(t,s)\|_2^2\,dtds \\ &\leq c\int_{\mathbb{R}^2}(1 - \varphi_n(t)\varphi_n(s))^2\|B(t)\|\,\|B(s)\||t-s|^{-\gamma}dtds \\ &\quad + c\delta_n^2\int_{\mathbb{R}^2}\|B(t)\|\,\|B(s)\|\,|t-s|^{-\gamma}dtds. \end{aligned}$$

From Lebesgue's dominated convergence theorem, the Hardy-Littlewood inequality (2.23), and condition (2.2), we conclude that $\mathcal{K}_n \to \mathcal{K}$ in $\mathcal{B}_2(L^2(\mathbb{R},\mathcal{X}))$ as $n \to \infty$.
Proof of (ii). Similarly to the proof of Lemma 2.3, one can show that the operator $(\mathcal{K}_n - \mathcal{K})^*(\mathcal{K}_n - \mathcal{K})$ has the integral kernel $L_n(t,\tau)$ defined by

$$L_n(t,\tau)x = \int_{\mathbb{R}}(K_n(s,t) - K(s,t))^*(K_n(s,\tau) - K(s,\tau))x\,ds, \quad x \in \mathcal{X},$$

and, moreover, $L_n(t,\tau) \in \mathcal{B}_2(\mathcal{X})$ for all $t \neq \tau$. Using Lemma A.1, the Riesz composition formula (2.24) with $d = 1$ and $\alpha = \gamma/2$, estimate (2.33), the inequality

$$(1 - \varphi_n(t)\varphi_n(s))(1 - \varphi_n(\tau)\varphi_n(s)) \leq (1 - \varphi_n(t))(1 - \varphi_n(\tau)) + 4(1 - \varphi_n(s)),$$

and letting $g_n(t,\tau) = \int_{\mathbb{R}}(1 - \varphi_n(s))|s-t|^{-\gamma/2}|s-\tau|^{-\gamma/2}ds$, for $t \neq \tau$ we obtain:

$$\begin{aligned} \|L_n(t,\tau)\|_2 &\leq \int_{\mathbb{R}}\|K_n(s,t)^* - K(s,t)^*\|_2\|K_n(s,\tau) - K(s,\tau)\|_2\,ds \\ &\leq \int_{\mathbb{R}}\|K_n(s,t) - K(s,t)\|_2\|K_n(s,\tau) - K(s,\tau)\|_2\,ds \\ &\leq c\|B(t)\|^{1/2}\|B(\tau)\|^{1/2}\int_{\mathbb{R}}|s-t|^{-\gamma/2}|s-\tau|^{-\gamma/2} \\ &\qquad \times \Big(c^2\delta_n^2 + 2c\delta_n + (1 - \varphi_n(t))(1 - \varphi_n(\tau)) + 4(1 - \varphi_n(s))\Big)ds \end{aligned}$$

$$\leq c\|B(t)\|^{1/2}\|B(\tau)\|^{1/2} \times \Big((\delta_n^2+\delta_n+(1-\varphi_n(t))(1-\varphi_n(\tau)))|t-\tau|^{1-\gamma}+4g_n(t,\tau)\Big).$$

From the Riesz composition formula (2.24) with $d=1$ and $\alpha=\gamma/2$ and Lebesgue's dominated convergence theorem it follows that $g_n(t,\tau)\to 0$ as $n\to\infty$ and $0\leq g_n(t,\tau)\leq c|t-\tau|^{1-\gamma}$ for all $t\neq\tau$. Since $\|B(\cdot)\|\in L^p(\mathbb{R})$, with $p=\frac{1}{2-\gamma}$ from Lebesgue's dominated convergence theorem and the Hardy-Littlewood inequality (2.23) we infer:

$$\int_{\mathbb{R}^2}\|B(t)\|\,\|B(\tau)\|g_n^2(t,\tau)\,dtd\tau\to 0 \quad\text{as}\quad n\to\infty. \tag{2.34}$$

Using formula (2.22), the estimate

$$\begin{aligned}\|\mathcal{K}_n-\mathcal{K}\|_4^4&=\|(\mathcal{K}_n-\mathcal{K})^*(\mathcal{K}_n-\mathcal{K})\|_2^2=\int_{\mathbb{R}^2}\|L_n(t,\tau)\|_2^2\,dtd\tau\\ &\leq c\delta_n\int_{\mathbb{R}^2}\|B(t)\|\,\|B(\tau)|t-\tau|^{1-\gamma}dtd\tau+c\int_{\mathbb{R}^2}\|B(t)\|\,\|B(\tau)\|g_n^2(t,\tau)\,dtd\tau\\ &\quad+c\int_{\mathbb{R}^2}(1-\varphi_n(t))(1-\varphi_n(\tau))\|B(t)\|\,\|B(\tau)\|\,|t-\tau|^{1-\gamma}dtd\tau,\end{aligned}$$

Lebesgue dominated convergence theorem, the Hardy-Littlewood inequality (2.23), and (2.34), the conclusion follows.

Proof of (iii). Using (2.32), (2.11) and the fact that $B_\ell(t),B_r(t)\in\mathcal{B}_4(\mathcal{X})$ and $\|B_\ell(t)\|_4=\|B_r(t)\|_4=\|B(t)\|_2^{1/2}$ for all $t\in\mathbb{R}$, we obtain

$$\begin{aligned}\|K_n(t,s)-K(t,s)\|_2&\leq(1-\varphi_n(t)\varphi_n(s))\|B_r(t)\mathcal{Q}^{(n)}\mathcal{V}(t-s)\mathcal{Q}^{(n)}B_\ell(s)\|_2\\ &\quad+\|B_r(t)\|_4\|\mathcal{V}_n(t-s)\|\|B_\ell(s)\|_4\\ &\leq(1-\varphi_n(t)\varphi_n(s))\|B(t)\|_2^{1/2}\|B(s)\|_2^{1/2}e^{-c|t-s|}\\ &\quad+\|B(t)\|_2^{1/2}\|\mathcal{V}_n(t-s)\|\|B(s)\|_2^{1/2}\leq h_n(t,s),\end{aligned}$$

where we denoted

$$h_n(t,s):=(1-\varphi_n(t)\varphi_n(s)+\|\mathcal{V}_n(t-s)\|)\|B(t)\|_2^{1/2}\|B(s)\|_2^{1/2}.$$

From (2.2) and (2.31) we have that $\|\mathcal{V}_n(t-s)\|\to 0$ as $n\to\infty$ for all $t\neq s$ and since $\varphi_n(t)\to 1$ as $n\to\infty$ for all $t\in\mathbb{R}$ we infer that $h_n(t,s)\to 0$ as $n\to\infty$ for all $t\neq s$. Moreover, for all $t,s\in\mathbb{R}$

$$h_n(t,s)\leq(1+\|\mathcal{V}_n(t-s)\|)\|B(t)\|_2^{1/2}\|B(s)\|_2^{1/2}\leq c\|B(t)\|_2^{1/2}\|B(s)\|_2^{1/2}.$$

Using Lebesgue dominated convergence theorem and (2.22) we conclude that $\mathcal{K}_n\to\mathcal{K}$ in $\mathcal{B}_2(L^2(\mathbb{R},\mathcal{X}))$ as $n\to\infty$.

Proof of (iv). It is similar to that of (ii) and Lemma 2.3(iv). □

Projections Q_j, $j\in\mathbb{Z}\setminus\{0\}$, define the exponential splitting for the unperturbed equation $u'(t)=Au(t)$. Next, we will show the existence of an exponential splitting for the perturbed equation $u'(t)=\big(A+B(t)\big)u(t)$. As we have mentioned, the *mild* solution of the unperturbed differential equation $u'(t)=Au(t)+f(t)$ on $\mathbb{R}$ is given by the convolution $\mathcal{V}*f$. We will now offer a convenient Fourier transform reformulation of the perturbed differential equation $u'(t)=\big(A+B(t)\big)u(t)+f(t)$ and define a related operator, G, which plays the role of the differential operator $\frac{d}{dt}-\big(A+B(t)\big)$ in our infinite–dimensional setting. In the finite dimensional setting, the kernel of this operator is non trivial if and only if the corresponding Evans determinant is equal to zero.

Note that $\sigma(A) \cap i\mathbb{R} = \emptyset$ due to (2.1). This fact allows us to define the function $R : \mathbb{R} \to \mathcal{B}(\mathcal{X})$ by $R(\xi) = R(2\pi i\xi, A)$, using the resolvent operator for A. Since $R(\lambda, A_\pm)x = \int_0^\infty e^{-\lambda t} T_\pm(t)x dt$ for all $x \in \mathcal{X}_\pm$, and $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda \geq 0$, we calculate:

$$\mathcal{F}\mathcal{V}(\cdot)x = R(\cdot)x \tag{2.35}$$

for all $x \in \mathcal{X}$, and hence $R(\cdot)x \in L^2(\mathbb{R}, \mathcal{X}) \cap C_0(\mathbb{R}, \mathcal{X})$ by the Riemann-Lebesgue Lemma. Similarly, $R(\cdot)^*x \in L^2(\mathbb{R}, \mathcal{X}) \cap C_0(\mathbb{R}, \mathcal{X})$ for all $x \in \mathcal{X}$. By the Closed Graph Theorem, for some $c > 0$ we have $\|R(\cdot)x\|_2 \leq c\|x\|$ and $\|R(\cdot)^*x\|_2 \leq c\|x\|$ for all $x \in \mathcal{X}$.

Further, we recall that $B : \mathbb{R} \to \mathcal{B}(\mathcal{X})$ is a piecewise strongly continuous operator valued function. In addition, we assume $\|B(\cdot)\| \in L^\infty(\mathbb{R})$, and introduce the operator G on $L^2(\mathbb{R}, \mathcal{X})$ as follows. Let $\operatorname{dom}(G)$ be the set of all $u \in L^2(\mathbb{R}, \mathcal{X})$ such that there exists $f \in L^2(\mathbb{R}, \mathcal{X})$ for which the relation

$$(\mathcal{F} - M_R\mathcal{F}M_B)u = M_R\mathcal{F}f \tag{2.36}$$

holds. Note that this f is unique by the injectivity of M_R, and define G by $Gu = f$ for $u \in \operatorname{dom}(G)$. Replacing the operator M_B with M_{B_n} we construct in a similar manner the operator G_n. Since, in general, $\mathcal{Q}_\pm$ and $B(t)$ do not commute (see examples in [10, 34]), the equation $u'(t) = (A+B(t))u(t)+f(t)$, where $f \in L^2(\mathbb{R}, \mathcal{X})$, is much harder to handle than its equivalent frequency domain reformulation (2.36).

Lemma 2.5. *Assume conditions* (2.1)–(2.2) *and* $\|B(\cdot)\| \in L^\infty(\mathbb{R})$. *If, in addition,* $\|B(\cdot)\| \in L^2(\mathbb{R})$ *or* $\lim_{|t|\to\infty} \|B(t)\| = 0$, *then G is Fredholm on* $L^2(\mathbb{R}, \mathcal{X})$ *and* $\operatorname{ind}(G) = 0$.

Proof. The proof closely follows [25, Prop. 7.3,7.4]. By [25, Prop. 7.2], it is enough to show the assertion $\mathcal{V} * M_B \in \mathcal{B}_\infty(\mathcal{X})$. Indeed, this essentially, follows from the identity $G = G_{\text{unp}}(I - \nu * M_B)$, where G_{unp} is the invertible operator corresponding to the unperturbed equation $u'(t) = Au(t)$, see [25, Section 7] for details.

Case 1. Assume that $\|B(\cdot)\| \in L^2(\mathbb{R})$. The integral kernel of the operator $\mathcal{V}*M_B$ is given by $S(t,s) = \mathcal{V}(t-s)B(s)$. Since $S(t,s) \in \mathcal{B}_2(\mathcal{X}) \subseteq \mathcal{B}_\infty(\mathcal{X})$ by Lemma 2.1, S is strongly measurable, $\|S(t,s)\| \leq e^{-c|t-s|}\|B(s)\|$ for all $t, s \in \mathbb{R}$, by (2.11) and $\int_{\mathbb{R}} \|B(t)\|^2\, dt < \infty$, one has $\int_{\mathbb{R}^2} \|S(t,s)\|^2\, dtds < \infty$, and thus [4, Prop. 2.1] gives the desired assertion $\mathcal{V} * M_B \in \mathcal{B}_\infty(\mathcal{X})$.

Case 2. Assume that $\lim_{|t|\to\infty} \|B(t)\| = 0$. Let $F_n : \mathbb{R} \to \mathcal{B}(X)$ be defined by $F_n(t) = \varphi_n(t)B(t)$ and let $\epsilon_n = \sup_{t\in\mathbb{R}} \|F_n(t) - B(t)\|$. Since $\epsilon_n \leq 2\sup_{|t|\geq n} \|B(t)\|$ we have $\epsilon_n \to 0$ as $n \to \infty$. We estimate

$$\begin{aligned}
&\|\Big((\mathcal{V} * M_{F_n} - \mathcal{V} * M_B)f\Big)(t)\| = \|\int_{\mathbb{R}} \mathcal{V}(t-s)(F_n(s) - B(s))f(s)\, ds\| \\
&\leq \int_{\mathbb{R}} e^{-c|t-s|}\|F_n(s) - B(s)\|\, \|f(s)\|\, ds \leq \epsilon_n \int_{\mathbb{R}} e^{-\frac{c}{2}|t-s|} \left(e^{-\frac{c}{2}|t-s|}\|f(s)\|\right) ds \\
&\leq c\epsilon_n \Big(\int_{\mathbb{R}} e^{-c|t-s|}\|f(s)\|^2\, ds\Big)^{1/2},
\end{aligned}$$

for all $n \in \mathbb{Z}_+$, $t \in \mathbb{R}$ and all $f \in L^2(\mathbb{R}, X)$. Integrating with respect to t, one has

$$\|(\mathcal{V} * M_{F_n} - \mathcal{V} * M_B)f\|_2 \leq c\epsilon_n \Big(\int_{\mathbb{R}}\int_{\mathbb{R}} e^{-c|t-s|}\|f(s)\|^2\, ds\, dt\Big)^{1/2} = c\epsilon_n\|f\|_2,$$

which implies that $\|\mathcal{V} * M_{F_n} - \mathcal{V} * M_B\| \to 0$ as $n \to \infty$. Since $\|F_n(\cdot)\| \in L^2(\mathbb{R})$, from Case 1 we conclude that $\mathcal{V} * M_{F_n} \in \mathcal{B}_\infty(\mathcal{X})$, proving the lemma. □

In the next sections, in addition to the operator valued generalized Jost solutions for the perturbed equation $u' = \big(A + B(t)\big)u(t)$, we will need to construct the generalized Jost solutions for its Galerkin approximation given in, equation (2.37) below.

Recall (2.25) and let $B_n : \mathbb{R} \to \mathcal{B}(\mathcal{X})$ be defined by $B_n(t) = \varphi_n^2(t)\mathcal{Q}^{(n)}B(t)\mathcal{Q}^{(n)}$. Consider the equation on $\mathcal{X}$,

$$u'(t) = \big(A + B_n(t)\big)u(t), \quad t \in \mathbb{R}. \tag{2.37}$$

Note that equation (2.37) decouples as follows,

$$u'(t) = \big(A\mathcal{Q}^{(n)} + B_n(t)\big)u(t), \quad t \in \mathbb{R}, \quad \text{in } \operatorname{im} \mathcal{Q}^{(n)}, \tag{2.38}$$

$$u'(t) = A(I - \mathcal{Q}^{(n)})u(t), \quad t \in \mathbb{R}, \quad \text{in } \ker \mathcal{Q}^{(n)}. \tag{2.39}$$

Using (2.1) we conclude that the equation

$$u'(t) = A\mathcal{Q}^{(n)}u(t), \quad t \in \mathbb{R}, \quad \text{in } \operatorname{im} \mathcal{Q}^{(n)}, \tag{2.40}$$

has exponential dichotomy on $\mathbb{R}$ with dichotomy projection $\mathcal{Q}_+^{(n)}$. Moreover, since $\dim \operatorname{im} \mathcal{Q}^{(n)} < \infty$ and $\|B_n(\cdot)\|$ has compact support for each n, using the classical results on persistence of the exponential dichotomy and the corresponding Lyapunov exponents under L^1-perturbations, cf. [6, Proposition 4.1] and [8, Theorem IV.5.1], we infer that equation (2.38) has exponential dichotomy on $\mathbb{R}_\pm$, in $\operatorname{im} \mathcal{Q}^{(n)}$, with dichotomy projections $\mathcal{P}_\pm^{(n)}$ which can be chosen such that $\ker \mathcal{P}_+^{(n)} = \ker \mathcal{Q}_+^{(n)}$ and $\operatorname{im} \mathcal{P}_-^{(n)} = \operatorname{im} \mathcal{Q}_+^{(n)} (= \ker \mathcal{Q}_-^{(n)})$. It follows that

$$\mathcal{Q}_+^{(n)} = \mathcal{Q}_+^{(n)}\mathcal{P}_+^{(n)} \text{ and } \mathcal{Q}_-^{(n)} = \mathcal{Q}_-^{(n)}\big(I_{\operatorname{im} \mathcal{Q}^{(n)}} - \mathcal{P}_-^{(n)}\big). \tag{2.41}$$

3. The generalized operator-valued Jost solutions

In this section we introduce and study the generalized operator-valued Jost solutions $Y_\pm^{(j)}$, $j \in \mathbb{Z} \setminus \{0\}$, on $\mathbb{R}_\pm$, of the perturbed differential equation $u'(t) = \big(A + B(t)\big)u(t)$ that are asymptotic to the solutions $T_\pm(\pm t)Q_j$ of the unperturbed equation $u'(t) = Au(t)$.

Definition 3.1. The generalized operator-valued Jost solutions are defined as solutions of the following Fredholm integral equations:

$$\begin{aligned} Y_+^{(j)}(t) = T_+(t)Q_j &- \int_t^\infty \Big[T_+(t-s)\big(\sum_{k=j}^{-1} Q_k\big) + T_-(s-t)\mathcal{Q}_-\Big]B(s)Y_+^{(j)}(s)\,ds \\ &+ \int_0^t T_+(t-s)\big(\sum_{k=-\infty}^{j-1} Q_k\big)B(s)Y_+^{(j)}(s)\,ds,\ t \ge 0,\ j \in \mathbb{Z}_-; \end{aligned} \tag{3.1}$$

$$\begin{aligned} Y_-^{(j)}(t) = T_-(-t)Q_j &+ \int_{-\infty}^t \Big[T_+(t-s)\mathcal{Q}_+ + T_-(s-t)\big(\sum_{k=1}^{j} Q_k\big)\Big]B(s)Y_-^{(j)}(s)\,ds \\ &- \int_t^0 T_-(s-t)\big(\sum_{k=j+1}^{\infty} Q_k\big)B(s)Y_-^{(j)}(s)\,ds,\ t \le 0,\ j \in \mathbb{Z}_+. \end{aligned} \tag{3.2}$$

We remark that $\operatorname{im}(\sum_{k=j}^{-1} Q_k)$ is finite dimensional, and thus the term $T_+(t-s)(\sum_{k=j}^{-1} Q_k)$ in (3.1) is well-defined although $t-s\le 0$ (and similarly $T_-(s-t)(\sum_{k=1}^{j} Q_k)$ in (3.2) is well-defined although $s-t\le 0$).

Throughout, we assume the following hypothesis.

Hypothesis 3.2. *The operator valued function $B:\mathbb{R}\to\mathcal{B}(\mathcal{X})$ is piecewise continuous $\|B(\cdot)\|\in L^\infty(\mathbb{R})$, and satisfies the following properties:*

(b_1) $B(t)\in\mathcal{B}_2(\mathcal{X})$ *for all* $t\in\mathbb{R}$*;*
(b_2) $\int_{\mathbb{R}_\pm}\|B(t)\|_2\,dt<1$.

It is an interesting open problem to replace the smallness condition (b_2) by a much weaker assumption $\|B(\cdot)\|_2\in L^1(\mathbb{R})$, see a discussion of this issue at the end of this section.

Next, we define $F_j^+ : C_{\rm b}([0,\infty),\mathcal{B}(\mathcal{X}))\to C_{\rm b}([0,\infty),\mathcal{B}_2(\mathcal{X}))$ by

$$(F_j^+Z)(t) = -\int_t^\infty \Lambda_j^+(s-t)B(s)Z(s)ds + \int_0^t \Gamma_j^+(t-s)B(s)Z(s)ds \tag{3.3}$$

for all $Z\in C_{\rm b}([0,\infty);\mathcal{B}(\mathcal{X}))$ and $t\ge 0$, where $\Lambda_j^+,\Gamma_j^+ : [0,\infty)\to\mathcal{B}(\mathcal{X})$ are defined as follows:

$$\Lambda_j^+(t) = e^{\varkappa_j t}\big[T_+(-t)(\sum_{k=j}^{-1} Q_k)+T_-(t)\mathcal{Q}_-\big],\quad \Gamma_j^+(t) = e^{-\varkappa_j t}T_+(t)(\sum_{k=-\infty}^{j-1} Q_k). \tag{3.4}$$

Using (2.3) and (2.13) we readily have that the operators $\Lambda_j^+(t)$ and $\Gamma_j^+(t)$ are diagon operators for any $t\ge 0$ and $j\in\mathbb{Z}_-$. We remark that for any sequence $(\zeta_k)_{k\in\mathbb{Z}}\subset\mathbb{C}$ and projections Q_k we have:

$$\|\sum_k \zeta_k Q_k\|_{\mathcal{B}(X)} = \sup_k|\zeta_k| \tag{3.5}$$

In particular, it follows from (2.12) that

$$\|\Lambda_j^+(t)\|_{\mathcal{B}(\mathcal{X})}\le 1 \quad\text{and}\quad \|\Gamma_j^+(t)\|_{\mathcal{B}(\mathcal{X})}\le 1 \quad\text{for all}\quad t\in\mathbb{R}_+. \tag{3.6}$$

Similarly, we define $F_j^- : C_{\rm b}((-\infty,0]),\mathcal{B}(\mathcal{X}))\to C_{\rm b}((-\infty,0],\mathcal{B}_2(\mathcal{X}))$ by

$$(F_j^-Z)(t) = \int_{-\infty}^t \Lambda_j^-(s-t)B(s)Z(s)ds - \int_t^0 \Gamma_j^-(t-s)B(s)Z(s)ds \tag{3.7}$$

for all $Z\in C_{\rm b}((-\infty,0];\mathcal{B}(\mathcal{X}))$ and $t\le 0$, where $\Lambda_j^-,\Gamma_j^- : (-\infty,0]\to\mathcal{B}(\mathcal{X})$ are defined as follows:

$$\Lambda_j^-(t) = e^{\varkappa_j t}\big[T_+(-t)\mathcal{Q}_+ + T_-(t)(\sum_{k=1}^{j} Q_k)\big],\quad \Gamma_j^-(t) = e^{-\varkappa_j t}T_-(-t)(\sum_{k=j+1}^{\infty} Q_k). \tag{3.8}$$

Using (3.5) again, one can easily check that

$$\|\Lambda_j^-(t)\|\le 1,\quad \|\Gamma_j^-(t)\|\le 1 \quad\text{for all}\quad t\in\mathbb{R}_-. \tag{3.9}$$

In addition to the functions $Y_\pm^{(j)}$, we will introduce the functions

$$Z_\pm^{(j)}(t) = e^{-\varkappa_j t}Y_\pm^{(j)}(t),\quad j\in\mathbb{Z}\setminus\{0\}, t\in\mathbb{R}_\pm \tag{3.10}$$

and consider the following system of equations for $Z_{\pm}^{(j)}(t)$:

$$Z_{+}^{(j)}(t) = e^{-\varkappa_j t}T_{+}(t)Q_j + \big(F_j^{+} Z_{+}^{(j)}\big)(t), \quad t \geq 0, j \in \mathbb{Z}_{-}; \tag{3.11}$$

$$Z_{-}^{(j)}(t) = e^{-\varkappa_j t}T_{-}(-t)Q_j + \big(F_j^{-} Z_{-}^{(j)}\big)(t), \quad t \leq 0, j \in \mathbb{Z}_{+}. \tag{3.12}$$

Lemma 3.3. *Assume* (2.1)-(2.2) *and Hypothesis 3.2. There exists a constant c so that, uniformly for all $j \in \mathbb{Z} \setminus \{0\}$, the following statements hold:*

(i) Equation (3.1)*, respectively,* (3.2) *has a unique operator-valued solution $Y_{+}^{(j)} \in C_{\rm b}([0,+\infty), \mathcal{B}_2(\mathcal{X}); e^{-\varkappa_j t})$, respectively, $Y_{-}^{(j)} \in C_{\rm b}((-\infty, 0], \mathcal{B}_2(\mathcal{X}); e^{-\varkappa_j t})$;*

(ii) Moreover, $\|Y_{\pm}^{(j)}(t)\| \leq c e^{t\varkappa_j}$ for all $\pm t \geq 0$ and $j \in \mathbb{Z}_{\mp}$;

(iii) Finally, with the $o(1)$-term independent on j,

$$e^{-\varkappa_j t}\|Y_{+}^{(j)}(t) - T_{+}(t)Q_j\|_2 = o(1) \ \text{ as } t \to +\infty,\ j \in \mathbb{Z}_{-};$$
$$e^{-\varkappa_j t}\|Y_{-}^{(j)}(t) - T_{-}(t)Q_j\|_2 = o(1) \ \text{ as } t \to -\infty,\ j \in \mathbb{Z}_{+}.$$

Proof. We will use (2.15)-(2.16) in combination with [14, Lem. 6.4] as in the proof of [14, Thm. 8.3]. We claim that there exits a constant c so that, uniformly for all $j \in \mathbb{Z}$, the following statements hold:

(i') Equation (3.11), respectively, (3.12) has a unique operator-valued solution $Z_{+}^{(j)} \in C_{\rm b}([0,+\infty), \mathcal{B}_2(\mathcal{X}))$, respectively, $Z_{-}^{(j)} \in C_{\rm b}((-\infty, 0], \mathcal{B}_2(\mathcal{X}))$;

(ii') Moreover, $\|Z_{+}^{(j)}\|_{C_{\rm b}([0,+\infty),\mathcal{B}(\mathcal{X}))} \leq c$ and $\|Z_{-}^{(j)}\|_{C_{\rm b}((-\infty,0],\mathcal{B}(\mathcal{X}))} \leq c$.

(iii') Finally, with the $o(1)$-term independent on j,

$$\|Z_{+}^{(j)}(t) - e^{-\varkappa_j t}T_{+}(t)Q_j\|_2 = o(1) \ \text{ as } t \to +\infty,\ j \in \mathbb{Z}_{-};$$
$$\|Z_{-}^{(j)}(t) - e^{-\varkappa_j t}T_{-}(t)Q_j\|_2 = o(1) \ \text{ as } t \to -\infty,\ j \in \mathbb{Z}_{+}.$$

Clearly, (i),(ii), (iii) in the lemma follows from (i'), (ii"), (iii'), respectively. We will give the proof of (i'), (ii"), (iii') for $j \in \mathbb{Z}_{-}$; the argument for $j \in \mathbb{Z}_{+}$ is similar.

One can readily check by (3.6) and Hypothesis 3.2 that

$$\|F_j^{+}\|_{\mathcal{B}(C_{\rm b}([0,+\infty),\mathcal{B}(\mathcal{X})))} \leq \int_0^\infty \|B(s)\|_2\, ds < 1. \tag{3.13}$$

Since

$$\sup_{t \geq 0} \|e^{-\varkappa_j t}T_{+}(t)Q_j\|_{\mathcal{B}(\mathcal{X})} = 1,$$

(i') and (ii') follow from (3.13).

Using that $\varkappa_{j-1} \geq \varkappa_k$ and $\varkappa_{j-1} < \varkappa_j$ whenever $k \leq j-1$, and (2.3), we obtain:

$$\|(F_j^{+} Z_{+}^{(j)})(t)\|_{\mathcal{B}_2(\mathcal{X})} \leq \|Z_{+}^{(j)}\|_{C_{\rm b}([0,\infty);\mathcal{B}(\mathcal{X}))} \Big(\int_t^\infty \|B(s)\|_2\, ds \tag{3.14}$$

$$+ \int_0^t e^{(\varkappa_{j-1} - \varkappa_j)(t-s)} \|B(s)\|_2\, ds \Big),\ t \geq 0. \tag{3.15}$$

Since $\|B(\cdot)\|_2 \in L^1(\mathbb{R})$ assertion (iii') now follows from (3.15) and [14, Lemma 6.4] with $\alpha = \varkappa_j - \varkappa_{j-1}$. □

We remark that $Y_{\pm}^{(j)}(t) = Y_{\pm}^{(j)}(t)Q_j$ for all $j \in \mathbb{Z}_{\mp}$ due to the uniqueness of solutions, and hence the values of the solutions $Y_{\pm}^{(j)}$ are operators of finite rank. Thus, one can easily check that the operator valued functions $Y_{\pm}^{(j)}(\cdot)$ are solutions of the perturbed equation $u'(t) = [A + B(t)]u(t)$, by differentiating equations (3.1) and (3.2), respectively.

In Theorem 3.6, we will prove that the series $\sum_{j\in\mathbb{Z}_\mp} Y_\pm^{(j)}(0)$ converge strongly. This will require some preparations. From (3.1) and (3.2) we obtain that

$$Y_+^{(j)}(0) = Q_j - \int_0^\infty \Big[T_+(-s)(\sum_{k=j}^{-1} Q_k) + T_-(s)\mathcal{Q}_-\Big]B(s)Y_+^{(j)}(s)\,ds, j \in \mathbb{Z}_-, \quad (3.16)$$

$$Y_-^{(j)}(0) = Q_j + \int_{-\infty}^0 \Big[T_+(-s)\mathcal{Q}_+ + T_-(s)(\sum_{k=1}^{j} Q_k)\Big]B(s)Y_-^{(j)}(s)\,ds\, j \in \mathbb{Z}_+. \quad (3.17)$$

In the next lemmas we will show that the operator $\sum_{j\in\mathbb{Z}_\mp} Y_\pm^{(j)}(0)$ on $\mathcal{X}$ can be written as an infinite–dimensional operator valued block–matrix with entries $L_{kj}^\pm$, defined below, see (3.28).

For any $\mathbf{Z} = \{Z^{(j)}\}_{j\le -1} \in \ell^\infty(\mathbb{Z}_-, C_{\rm b}(\mathbb{R}_+, \mathcal{B}(\mathcal{X})))$ we define the linear operators $L_{kj}^+(B, \mathbf{Z}) : \mathcal{X} \to \mathcal{X}$ by

$$\begin{aligned} L_{kj}^+(B,\mathbf{Z})x &= \int_0^\infty e^{\varkappa_j s} T_+(-s)Q_k B(s)Z^{(j)}(s)x\,ds \quad \text{for} \quad j \le k \le -1,\\ L_{kj}^+(B,\mathbf{Z})x &= \int_0^\infty e^{\varkappa_j s} T_-(s)Q_k B(s)Z^{(j)}(s)x\,ds \quad \text{for} \quad k \ge 1,\, j \le -1. \end{aligned} \quad (3.18)$$

Similarly, for any $\mathbf{Z} = \{Z^{(j)}\}_{j\ge 1} \in \ell^\infty(\mathbb{Z}_+, C_{\rm b}(\mathbb{R}_-, \mathcal{B}(\mathcal{X})))$ we define the linear operators $L_{kj}^-(B, \mathbf{Z}) : \mathcal{X} \to \mathcal{X}$ by

$$\begin{aligned} L_{kj}^-(B,\mathbf{Z})x &= \int_{-\infty}^0 e^{\varkappa_j s} T_-(s)Q_k B(s)Z^{(j)}(s)x\,ds \quad \text{for} \quad j \ge k \ge 1,\\ L_{kj}^-(B,\mathbf{Z})x &= \int_{-\infty}^0 e^{\varkappa_j s} T_+(-s)Q_k B(s)Z^{(j)}(s)x\,ds \quad \text{for} \quad k \le -1,\, j \ge 1. \end{aligned} \quad (3.19)$$

Using the definition of the projections Q_k, $k \in \mathbb{Z}\setminus\{0\}$ we infer that

$$\|T_\pm(\mp s)Q_k\| = e^{-\varkappa_k s} \quad \text{for all} \quad s \in \mathbb{R},\, k \in \mathbb{Z}\setminus\{0\}. \quad (3.20)$$

This equality and Lemma A.1 imply that the linear operators $L_{jj}^\pm(B,\mathbf{Z})$ belong to $\mathcal{B}_2(\mathcal{X})$ and

$$\|L_{jj}^\pm(B,\mathbf{Z})\|_2 \le \|\mathbf{Z}\|_\infty \int_{\mathbb{R}_\pm} \|B(s)\|_2 ds, \quad (3.21)$$

for all $j \in \mathbb{Z}_\mp$ and $\mathbf{Z} \in \ell^\infty(\mathbb{Z}_\mp, C_{\rm b}(\mathbb{R}_\pm, \mathcal{B}(\mathcal{X})))$. Moreover, using again (3.20) and Lemma A.1 we conclude that the finite rank operators $L_{kj}^\pm(B,\mathbf{Z})$ satisfy

$$\begin{aligned} \|L_{kj}^\pm(B,\mathbf{Z})\|_2 &\le \|Z^{(j)}\|_{\mathcal{C}_{\rm b}^\pm} \int_{\mathbb{R}_\pm} e^{(\varkappa_j-\varkappa_k)s}\|B(s)\|_2\,ds\\ &\le \sup_{s\in\mathbb{R}} \|B(s)\|_2 \|\mathbf{Z}\|_\infty \int_{\mathbb{R}_\pm} e^{(\varkappa_j-\varkappa_k)s} ds\\ &= \|\mathbf{Z}\|_\infty \sup_{s\in\mathbb{R}} \|B(s)\|_2 |\varkappa_k - \varkappa_j|^{-1}, \quad k \ne j, \end{aligned} \quad (3.22)$$

for all $(k,j) \in \mathbb{Z}_{2,\pm}$ and $\mathbf{Z} \in \ell^\infty(\mathbb{Z}_\mp, C_{\rm b}(\mathbb{R}_\pm, \mathcal{B}(\mathcal{X})))$. Here the sets $Z_{2,\pm}$ are defined in Hypothesis 2.2.

In the sequel, it is convenient to decompose the operator-valued function B as a product of a scalar L^2 function and an operator valued function as follows. Let us define $b:\mathbb{R}\to\mathbb{R}_+$ and $\tilde{B}:\mathbb{R}\to\mathcal{B}_2(\mathcal{X})$ by

$$b(t)=\|B(t)\|_2^{1/2}\quad\text{and}\quad \tilde{B}(t)=\begin{cases}\|B(t)\|_2^{-1/2}B(t), & \text{if } B(t)\neq 0,\\ 0, & \text{if } B(t)=0.\end{cases}\tag{3.23}$$

Since $B(t)\in\mathcal{B}_2(\mathcal{X})$ for all $t\in\mathbb{R}$ and $\|B(\cdot)\|_2\in L^1(\mathbb{R})$ we have $b\in L^2(\mathbb{R})$, $\tilde{B}(t)\in\mathcal{B}_2(\mathcal{X})$ for all $t\in\mathbb{R}$, $\|\tilde{B}(\cdot)\|_2\in L^2(\mathbb{R})$ and

$$B(t)=b(t)\tilde{B}(t)\quad\text{and}\quad t\in\mathbb{R}.\tag{3.24}$$

Recall notation N_j introduced in (2.13), and the definition of the operators $L^{\pm}_{kj}$ in (3.18)–(3.19).

Lemma 3.4. *Assume Hypotheses 2.2 and 3.2. For any $\mathbf{Z}\in\ell^\infty(\mathbb{Z}_\mp,C_{\rm b}(\mathbb{R}_\pm,\mathcal{B}(\mathcal{X})))$ satisfying the condition $Z^{(j)}(s)Q_j=Z^{(j)}(s)$ for all $s\in\mathbb{R}_\pm$, $j\in\mathbb{Z}_\mp$, we have:*

(i) *The linear operator $L^{\pm}_0(B,\mathbf{Z}):\mathcal{X}\to\mathcal{X}$ defined by*

$$L^{\pm}_0(B,\mathbf{Z})x=\sum_{j\in\mathbb{Z}_\mp}\sum_{m\in N_j}\sum_{n\in N_j}\langle x,v_n\rangle\langle L^{\pm}_{jj}(B,\mathbf{Z})v_n,v_m\rangle v_m\tag{3.25}$$

is well–defined and is Hilbert–Schmidt, $L^{\pm}_0(B,\mathbf{Z})\in\mathcal{B}_2(\mathcal{X})$;

(ii) *The following $\mathcal{B}_2$–estimate holds:*

$$\begin{aligned}\|L^{\pm}_0(B,\mathbf{Z})\|_2^2&\le\left(\int_{\mathbb{R}_\pm}\|B(s)\|_2\,ds\right)\sum_{j\in\mathbb{Z}_\mp}\sum_{m\in N_j}\|Z^{(j)}\|^2_{\mathcal{C}^\pm_{\rm b}}\left(\int_{\mathbb{R}_\pm}\|\tilde{B}(s)^*v_m\|^2ds\right)\\&\le\|\mathbf{Z}\|^2_\infty\left(\int_{\mathbb{R}_\pm}\|B(s)\|_2\,ds\right)^2,\end{aligned}$$

where $\tilde{B}(\cdot)$ is defined in (3.23);

(iii) *The following strong convergence holds true:*

$$\sum_{j\in\mathbb{Z}_\mp}L^{\pm}_{jj}(B,\mathbf{Z})x=L^{\pm}_0(B,\mathbf{Z})x\quad\textit{for each}\quad x\in\mathcal{X}.$$

Proof. Throughout the proof we abbreviate $L^{\pm}_0:=L^{\pm}_0(B,\mathbf{Z})$ and $L^{\pm}_{jj}:=L^{\pm}_{jj}(B,\mathbf{Z})$, $j\in\mathbb{Z}_\mp$.

Proof of (i) and (ii). Using Parseval's identity and (3.21), we estimate

$$\begin{aligned}&\sum_{j\in\mathbb{Z}_\mp}\sum_{m\in N_j}\Big|\sum_{n\in N_j}\langle x,v_n\rangle\langle L^{\pm}_{jj}v_n,v_m\rangle\Big|^2=\sum_{j\in\mathbb{Z}_\mp}\sum_{m\in N_j}\Big|\langle L^{\pm}_{jj}Q_jx,v_m\rangle\Big|^2\\&\le\sup_{j\in\mathbb{Z}_\mp}\|L^{\pm}_{jj}\|^2\sum_{j\in\mathbb{Z}_\mp}\|Q_jx\|^2\le c\|\mathbf{Z}\|^2_\infty\Big(\int_{\mathbb{R}_\pm}\|B(s)\|_2ds\Big)^2\sum_{j\in\mathbb{Z}_\mp}\|Q_jx\|^2\\&=c\|\mathbf{Z}\|^2_\infty\Big(\int_{\mathbb{R}_\pm}\|B(s)\|_2ds\Big)^2\|\mathcal{Q}_\pm x\|^2\le c\|\mathbf{Z}\|^2_\infty\Big(\int_{\mathbb{R}_\pm}\|B(s)\|_2ds\Big)^2\|x\|^2<\infty.\end{aligned}$$

It follows that $L^{\pm}_0$ is well-defined and bounded. To prove that $L^{\pm}_0\in\mathcal{B}_2(\mathcal{X})$ we will show that $\sum_{n\in\mathbb{Z}\setminus\{0\}}\|L^{\pm}_0v_n\|^2<\infty$. First, we note that for any $n\in\mathbb{Z}\setminus\{0\}$

there exists a unique $j \in \mathbb{Z}_\pm$ such that $n \in N_j$. Since $Z^{(j)}(s)Q_j = Z^{(j)}(s)$ for all $s \in \mathbb{R}_\pm, j \in \mathbb{Z}_\mp$, we have

$$\|L_0^\pm v_n\|^2 = \sum_{m \in N_j} |\langle L_{jj}^\pm v_n, v_m\rangle|^2.$$

Using the definition of the operators $L_{jj}^\pm$ in (3.18)–(3.19) and (3.24), we estimate

$$\begin{aligned}
\sum_{n\in\mathbb{Z}\setminus\{0\}} \|L_0^\pm v_n\|^2 &= \sum_{j\in\mathbb{Z}_\mp}\sum_{n,m\in N_j} |\langle L_{jj}^\pm v_n, v_m\rangle|^2 \\
&= \sum_{j\in\mathbb{Z}_\mp}\sum_{n,m\in N_j} \left|\langle \int_{\mathbb{R}_\pm} e^{\varkappa_j s} T_\pm(\mp s) B(s) Z^{(j)}(s) v_n ds, v_m\rangle\right|^2 \\
&= \sum_{j\in\mathbb{Z}_\mp}\sum_{n,m\in N_j} \left|\int_{\mathbb{R}_\pm} e^{\varkappa_j s} \langle b(s) T_\pm(\mp s) \tilde{B}(s) Z^{(j)}(s) v_n, v_m\rangle ds\right|^2 \\
&= \sum_{j\in\mathbb{Z}_\mp}\sum_{n,m\in N_j} \left|\int_{\mathbb{R}_\pm} e^{\varkappa_j s} b(s) \langle Z^{(j)}(s) v_n, \tilde{B}(s)^* T_\pm(\mp s)^* v_m\rangle ds\right|^2 \\
&= \sum_{j\in\mathbb{Z}_\mp}\sum_{n,m\in N_j} \left|\int_{\mathbb{R}_\pm} e^{(\varkappa_j \pm a_m)s} b(s) \langle Z^{(j)}(s) v_n, \tilde{B}(s)^* v_m\rangle ds\right|^2 \\
&\le \sum_{j\in\mathbb{Z}_\mp}\sum_{n,m\in N_j} \left(\int_{\mathbb{R}_\pm} b(s) \big|\langle v_n, Z^{(j)}(s)^* \tilde{B}(s)^* v_m\rangle\big| ds\right)^2 \\
&\le \sum_{j\in\mathbb{Z}_\mp}\sum_{n,m\in N_j} \left(\int_{\mathbb{R}_\pm} b^2(s)\, ds\right)\left(\int_{\mathbb{R}_\pm} \big|\langle v_n, Z^{(j)}(s)^* \tilde{B}(s)^* v_m\rangle\big|^2 ds\right) \\
&= \|b\|_2^2 \sum_{j\in\mathbb{Z}_\mp}\sum_{m\in N_j} \left(\int_{\mathbb{R}_\pm} \sum_{n\in N_j} \big|\langle v_n, Z^{(j)}(s)^* \tilde{B}(s)^* v_m\rangle\big|^2 ds\right) \\
&\le \|b\|_2^2 \sum_{j\in\mathbb{Z}_\mp}\sum_{m\in N_j} \int_{\mathbb{R}_\pm} \|Z^{(j)}(s)^* \tilde{B}(s)^* v_m\|^2 ds \\
&\le \left(\int_{\mathbb{R}_\pm} \|B(s)\|_2\, ds\right) \sum_{j\in\mathbb{Z}_\mp}\sum_{m\in N_j} \|Z^{(j)}\|^2_{\mathcal{C}_\mathrm{b}^\pm} \int_{\mathbb{R}_\pm} \|\tilde{B}(s)^* v_m\|^2 ds.
\end{aligned}$$

This proves the first inequality in (ii). To prove the second inequality we continue to estimate

$$\begin{aligned}
\sum_{n\in\mathbb{Z}\setminus\{0\}} \|L_0^\pm v_n\|^2 &\le \left(\int_{\mathbb{R}_\pm} \|B(s)\|_2\, ds\right) \|\mathbf{Z}\|_\infty^2 \int_{\mathbb{R}_\pm} \sum_{j\in\mathbb{Z}_\mp}\sum_{m\in N_j} \|\tilde{B}(s)^* v_m\|^2 ds \\
&\le \left(\int_{\mathbb{R}_\pm} \|B(s)\|_2\, ds\right) \|\mathbf{Z}\|_\infty^2 \int_{\mathbb{R}_\pm} \|\tilde{B}(s)^*\|_2^2 ds \\
&= \|\mathbf{Z}\|_\infty^2 \left(\int_{\mathbb{R}_\pm} \|B(s)\|_2\, ds\right)^2 < \infty,
\end{aligned}$$

thus proving (i) and (ii).

Proof of (iii). Using $L_{jj}^{\pm}=Q_jL_{jj}^{\pm}Q_j$ for all $j\in\mathbb{Z}_{\mp}$, see (3.18)–(3.19), we infer

$$L_{jj}^{\pm}x=\sum_{n,m\in N_j}\langle x,v_n\rangle\langle L_{jj}^{\pm}v_n,v_m\rangle v_m$$

for all $j\in\mathbb{Z}_{\mp}$. Convergence in (iii) follows from (i). □

Next, we prove that the sum $\sum_{(k,j)\in\mathbb{Z}_{2,\pm}}L_{kj}^{\pm}(B,\mathbf{Z})$ is strongly convergent and defines a Hilbert-Schmidt operator on $\mathcal{X}$ (see (2.17) for the definition of $\mathbb{Z}_{2,\pm}$).

Lemma 3.5. *Assume Hypotheses 2.2 and 3.2. For any $\mathbf{Z}\in\ell^{\infty}(\mathbb{Z}_{\mp},C_{\mathrm{b}}(\mathbb{R}_{\pm},\mathcal{B}(\mathcal{X})))$ satisfying the condition $Z^{(j)}(s)Q_j=Z^{(j)}(s)$ for all $s\in\mathbb{R}_{\pm}$, $j\in\mathbb{Z}_{\mp}$, we have:*

(i) *The linear operator $L_S(B,\mathbf{Z}):\mathcal{X}\to\mathcal{X}$ defined by*

$$L_S^{\pm}(B,\mathbf{Z})x=\sum_{(k,j)\in\mathbb{Z}_{2,\pm}}\sum_{m\in N_k}\sum_{n\in N_j}\langle x,v_n\rangle\langle L_{kj}^{\pm}(B,\mathbf{Z})v_n,v_m\rangle v_m \tag{3.26}$$

is well–defined and is Hilbert-Schmidt, $L_S^{\pm}(B,\mathbf{Z})\in\mathcal{B}_2(\mathcal{X})$;

(ii) *The following $\mathcal{B}_2$–estimate holds:*

$$\|L_S^{\pm}(B,\mathbf{Z})\|_2^2\le\sum_{(k,j)\in\mathbb{Z}_{2,\pm}}\|L_{kj}^{\pm}\|_2^2\le c\|\mathbf{Z}\|_{\infty}^2\sup_{s\in\mathbb{R}}\|B(s)\|_2^2;$$

(iii) *The following strong convergence holds true:*

$$\sum_{(k,j)\in\mathbb{Z}_{2,\pm}}L_{kj}^{\pm}(B,\mathbf{Z})x=L_0^{\pm}(B,\mathbf{Z})x\quad\textit{for each}\quad x\in\mathcal{X}.$$

Proof. The proof follows closely the proof of Lemma 3.4. We abbreviate again $L_S^{\pm}:=L_S^{\pm}(B,\mathbf{Z})$ and $L_{kj}^{\pm}:=L_{kj}^{\pm}(B,\mathbf{Z})$, $(k,j)\in\mathbb{Z}_{2,\pm}$.

Proof of (i) and (ii). First, we prove that the operator $L_S^{\pm}$ is well defined and bounded. Indeed, using (3.22), we estimate

$$\begin{aligned}\sum_{(k,j)\in\mathbb{Z}_{2,\pm}}\sum_{m\in N_k}\Big|\sum_{n\in N_j}\langle x,v_n\rangle\langle L_{kj}^{\pm}v_n,v_m\rangle\Big|^2&=\sum_{(k,j)\in\mathbb{Z}_{2,\pm}}\sum_{m\in N_k}\Big|\langle L_{kj}^{\pm}Q_jx,v_m\rangle\Big|^2\\&=\sum_{(k,j)\in\mathbb{Z}_{2,\pm}}\|L_{kj}^{\pm}Q_jx\|^2\le\sum_{(k,j)\in\mathbb{Z}_{2,\pm}}\|L_{kj}^{\pm}\|_2^2\,\|Q_jx\|^2\\&\le\Big(\sum_{(k,j)\in\mathbb{Z}_{2,\pm}}(\varkappa_k-\varkappa_j)^{-2}\Big)\|x\|<\infty.\end{aligned}$$

To show the $\mathcal{B}_2$–estimate in (ii) we show first that $\sum_{n\in\mathbb{Z}\setminus\{0\}}\|L_S^{\pm}v_n\|^2<\infty$. Using the definition of $L_{kj}^{\pm}$ we obtain:

$$\begin{aligned}\sum_{n\in\mathbb{Z}\setminus\{0\}}\|L_S^{\pm}v_n\|^2&=\sum_{(k,j)\in\mathbb{Z}_{2,\pm}}\sum_{m\in N_k}\sum_{n\in N_j}\Big|\langle L_{kj}^{\pm}v_n,v_m\rangle\Big|^2\\&=\sum_{(k,j)\in\mathbb{Z}_{2,\pm}}\sum_{n\in N_j}\sum_{m\in N_k}\Big|\langle L_{kj}^{\pm}v_n,v_m\rangle\Big|^2\\&\le\sum_{(k,j)\in\mathbb{Z}_{2,\pm}}\sum_{n\in N_j}\|L_{kj}^{\pm}v_n\|^2\end{aligned}$$

$$\leq \sum_{(k,j)\in\mathbb{Z}_{2,\pm}} \|L^{\pm}_{kj}\|_2^2.$$

To finish the proof of (ii) we use (3.22) to estimate

$$\sum_{(k,j)\in\mathbb{Z}_{2,\pm}} \|L^{\pm}_{kj}\|_2^2 \leq \sum_{(k,j)\in\mathbb{Z}_{2,\pm}} \Big(\|\mathbf{Z}\|_\infty^2 \sup_{s\in\mathbb{R}} \|B(s)\|_2^2 (\varkappa_k - \varkappa_j)^{-2}\Big)$$
$$= \|\mathbf{Z}\|_\infty^2 \sup_{s\in\mathbb{R}} \|B(s)\|_2^2 \Big(\sum_{(k,j)\in\mathbb{Z}_{2,\pm}} (\varkappa_k - \varkappa_j)^{-2}\Big).$$

Proof of (iii). Since $L^{\pm}_{kj} = Q_k L^{\pm}_{kj} Q_j$ for all $(k,j) \in \mathbb{Z}_{2,\pm}$, we infer

$$L^{\pm}_{kj} x = \sum_{m\in N_k} \sum_{n\in N_j} \langle x, v_n\rangle \langle L^{\pm}_{kj}(B,\mathbf{Z}) v_n, v_m\rangle v_m$$

for all $(k,j) \in \mathbb{Z}_{2,\pm}$. Convergence in (iii) follows from (i). □

In the next theorem we use previous lemmas to prove the main result of this section, that will alow us to define the *Evans determinant* as a modified 2–determinant, see (3.7) below.

Theorem 3.6. *Assume Hypotheses 2.2 and 3.2. Then the generalized operator valued Jost solutions* $Y^{(j)}_{\pm}$ *of* (3.1), (3.2) *have the following properties:*

(i) *The following series are strongly convergent*

$$\mathcal{Y}_\pm := \sum_{j\in\mathbb{Z}_\mp} Y^{(j)}_{\pm}(0); \tag{3.27}$$

(ii) *If* $\mathbf{Z}_\pm = \{Z^{(j)}_\pm\}_{j\in\mathbb{Z}_\mp}$ *are defined by* (3.11) *and* (3.12), $L^\pm_0$ *and* $L^\pm_S$ *are defined in Lemma 3.4 and Lemma 3.5, then*

$$\mathcal{Y}_\pm = \mathcal{Q}_\pm \mp L^\pm_0(B,\mathbf{Z}_\pm) \mp L^\pm_S(B,\mathbf{Z}_\pm);$$

(iii) $\mathcal{Y}_\pm - \mathcal{Q}_\pm \in \mathcal{B}_2(\mathcal{X})$.

Proof. By (3.16), (3.17) and (3.18), (3.19) we have

$$\begin{aligned} Y^{(j)}_{+}(0) &= Q_j - L^{+}_{jj}(B,\mathbf{Z}_+) - \sum_{k>j} L^{+}_{kj}(B,\mathbf{Z}_+) \quad \text{for all} \quad j\in\mathbb{Z}_-;\\ Y^{(j)}_{-}(0) &= Q_j + L^{-}_{jj}(B,\mathbf{Z}_-) + \sum_{k<j} L^{-}_{kj}(B,\mathbf{Z}_-) \quad \text{for all} \quad j\in\mathbb{Z}_+. \end{aligned} \tag{3.28}$$

Since the series $\mathcal{Q}_\pm = \sum_{j\in\mathbb{Z}_\mp} Q_j$ converges strongly, assertions (i) and (ii) follow from Lemma 3.4 (iii) and Lemma 3.5 (iii). To finish the proof of the theorem, we note that (iii) follows from (ii) and Lemma 3.4 (i) and Lemma 3.5 (i). □

Definition 3.7. Let $Y^{(j)}_\pm(\cdot)$, $j \in \mathbb{Z}_\mp$ be the generalized operator valued Jost solutions of the perturbed differential equation $u'(t) = (A + B(t))u(t)$, and define $\mathcal{Y}_\pm$ by (3.27). We define the *Evans determinant* as follows:

$$E := \det\nolimits_{2,\mathcal{X}}(\mathcal{Y}_+ + \mathcal{Y}_-). \tag{3.29}$$

The 2–modified Fredholm determinant in (3.29) is well-defined since by Theorem 3.6 we have:

$$\mathcal{Y}_+ + \mathcal{Y}_- - I = L_0^-(B, \mathbf{Z}_-) + L_S^-(B, \mathbf{Z}_-) - L_0^+(B, \mathbf{Z}_+) - L_S^+(B, \mathbf{Z}_+) \in \mathcal{B}_2(\mathcal{X}).$$

In the next sections we will relate the Evans determinant to the 2–modified Fredholm determinant $\det_{2,L^2(\mathbb{R},\mathcal{X})}(I - \mathcal{K})$, where $\mathcal{K}$ is the integral operator with the kernel defined in (2.20).

Discussion. We conclude this section by discussing replacing the smallness assumption (b_2) in Hypothesis 3.2 by the following weaker assumption.

Hypothesis 3.8. *The operator valued function $B : \mathbb{R} \to \mathcal{B}(\mathcal{X})$ is piecewise continuous $\|B(\cdot)\| \in L^\infty(\mathbb{R})$, and satisfies the following properties:*

- (b_1) *$B(t) \in \mathcal{B}_2(\mathcal{X})$ for all $t \in \mathbb{R}$;*
- (b_2) *$\int_{\mathbb{R}} \|B(t)\|_2 \, \mathrm{d}t < \infty$.*

The first main issue here is to prove the existence of the Jost solutions satisfying the uniform in j estimates as in Lemma 3.3. Indeed, assuming Hypothesis 3.8, and replacing the integrals $\int_0^t$ and $\int_t^0$ in (3.1), (3.2) by the integrals $\int_\tau^t$ and $\int_t^{-\tau}$ with some $\tau > 0$, one can extend the proof of Lemma 3.3 by replacing the integral $\int_0^\infty$ in (3.13) by the integral $\int_\tau^\infty$ for a large enough τ. As a result, the conclusions of Lemma 3.3 hold provided $[0, \infty)$ and $(-\infty, 0]$ are replaced by $[\tau, \infty)$ and $(-\infty, -\tau]$ with a large enough τ. However, the construction of the generalized operator valued Jost solutions in Lemma 3.3 was just the first step in building the operators $\mathcal{Y}_\pm$ in formula (3.27) of Theorem 3.6. For the latter, one would like to extend the Jost solutions $Y_\pm^{(j)}$ from $[\tau, \infty)$ and $(-\infty, -\tau]$ to $\mathbb{R}_\pm$, and, at the time of this writing, we were not able to achieve this with a suitable *uniform* norm control under the weaker than (b_2) assumption mentioned in Hypothesis 3.8.

The smallness condition $\int_{\mathbb{R}_\pm} ||B(t)||_2 dt < 1$ is used mainly in Lemma 3.3 to define the generalized operator–valued Jost solutions and most notably to show that $e^{-\chi_j t}\|Y_\pm^{(j)}(t)\|$ has a bound on $\mathbb{R}_\pm$, uniform in $j \in \mathbb{Z}_\mp$. The usual dynamical systems methods used to prove the dichotomy of the perturbed equation $u'(t) = (A + B(t))u(t)$ by Peterhof, Sandstede and Scheel in [30], and by the authors in [25], do not seem to work in proving the existence of the Jost solutions. The main difference between the objectives in [30] and in the current paper is as follows: The vector valued solutions $x^s(\cdot, 0, z)$ from [30] on $[0, \infty)$ are the (mild) solutions of an initial value problem with given data at $t = 0$, while the Jost solutions on $[0, \infty)$ in the current paper are the (mild) operator–valued solutions with the data prescribed at $+\infty$ by the conditions of Lemma 3.3(iii). The difficulty of defining the Jost solutions comes not only from the ill–posedness of the equation, but also from the fact that in infinite dimensional spaces solutions cannot be propagated backwards on $[0, \infty)$ from $[\tau, \infty)$ without assuming extra conditions.

We will now outline a possible plan of constructing the Jost solutions when Hypothesis 3.2 is replaced by Hypothesis 3.8. This task is beyond the scope of this paper and can be a major undertaking, as there are several issues, pointed out below, that one needs to address. Similarly to the definition of the linear operators F_j^+ in (3.3), we define the linear operators $G_j^+ : C_{\mathrm{b}}(\mathbb{R}_+, \mathcal{X}) \to C_{\mathrm{b}}(\mathbb{R}_+, \mathcal{X})$ by letting

$$(G_j^+ z)(t) = -\int_t^\infty \Lambda_j^+(s-t)B(s)z(s)ds + \int_0^t \Gamma_j^+(t-s)B(s)z(s)ds \qquad (3.30)$$

for all $z \in C_{\mathrm{b}}(\mathbb{R}_+, \mathcal{X})$ and $t \geq 0$. Next, we define the linear operators $\mathcal{G}^+ : \ell^\infty(\mathbb{Z}_-, C_{\mathrm{b}}(\mathbb{R}_+, \mathcal{X})) \to \ell^\infty(\mathbb{Z}_-, C_{\mathrm{b}}(\mathbb{R}_+, \mathcal{X}))$ by

$$\mathcal{G}^+(z_j)_{j\in\mathbb{Z}_-} = (G_j^+ z_j)_{j\in\mathbb{Z}_-}. \tag{3.31}$$

The main reason why one needs to work on the space $\ell^\infty(\mathbb{Z}_-, C_{\mathrm{b}}(\mathbb{R}_+, \mathcal{X}))$ rather than on $C_{\mathrm{b}}(\mathbb{R}_+, \mathcal{X})$, is that it is imperative to obtain the estimates for the Jost solutions that are *uniform* in $j \in \mathbb{Z}_-$.

Following the approach from [30], we will now prove that the linear operator $I - \mathcal{G}^+$ is Fredholm with index 0. For the proof of this theorem one needs a representation of the form "small+compact" for $\mathcal{G}^+$, and several technical auxiliary lemmas discussed at length in Appendix B.

Theorem 3.9. *Assume* (2.1) – (2.2) *and Hypothesis 3.8. Then, the linear operator* $I - \mathcal{G}^+$ *on* $\ell^\infty(\mathbb{Z}_-, C_{\mathrm{b}}(\mathbb{R}_+, \mathcal{X}))$ *is Fredholm with index* 0.

Proof. The theorem follows from (B.5), Lemma B.2, Lemma B.3, Lemma B.4 and Lemma B.10. □

This is a step in constructing the Jost solutions but much remains to be done. Our preliminary work shows that there exist finitely many $j \in \mathbb{Z}_-$ for which $\ker(I - G_j^+)$ is not necessarily trivial. We mention several related issues to be addressed:

(1) We do not know if $(e^{-\varkappa_j \cdot} T_+(\cdot) Q_j x)_{j\in\mathbb{Z}_-} \in \mathrm{im}(I - \mathcal{G}^+)$ for all $x \in \mathcal{X}$.
(2) Since $I - \mathcal{G}^+$ is not invertible, at the moment we cannot prove the required uniform in j bounds on the Jost solutions.
(3) Extending the Jost solutions from $[\tau, \infty)$ to $\mathbb{R}_+$ would require, in addition to Hypothesis 3.8, a backward/forward uniqueness condition similar to hypothesis (H5) in [30].
(5) Since the Jost solutions might not be unique, we will need to prove that the modified Evans determinant $\det_{2,\mathcal{X}}(\mathcal{Y}_+ + \mathcal{Y}_-)$ is independent of the choice of the Jost solutions. Arguing like in [14], one would need to prove that any element from $\ker(I - G_j^+)$ decays faster than $e^{\varkappa_j t}$ at $+\infty$ (and a similar result at $-\infty$) and that $\mathcal{Y}_+ + \mathcal{Y}_-$ has a certain block-diagonal structure.

In the current paper we choose not to pursue these issues and not to attempt to prove the existence of the Jost solutions in the most general case. Instead, we focus on constructing an infinite dimensional Evans function in the case when the Jost solutions do exist, that is, assuming Hypothesis 3.2.

4. Convergence of the 2-Modified Evans Determinants

In this section we show how the Evans determinant E, defined in (3.29) can be approximated by a sequence of 2-modified determinants E_n, on the finite dimensional subspace $\mathrm{im}\, \mathcal{Q}^{(n)}$, $n \in \mathbb{Z}_+$. Throughout this section we assume Hypotheses 2.2 and 3.2.

We will consider the following system of the truncated Fredholm integral equations:

$$\begin{aligned} Y_{+,n}^{(j)}(t) = T_+(t) Q_j \mathcal{Q}_+^{(n)} - \int_t^\infty \Big[& T_+(t-s)\mathcal{Q}_+^{(n)}(\sum_{k=j}^{-1} Q_k) \\ & + T_-(s-t)\mathcal{Q}_-^{(n)} \Big] B_n(s) Y_{+,n}^{(j)}(s)\, ds \end{aligned} \tag{4.1}$$

$$+\int_0^t T_+(t-s)\mathcal{Q}_+^{(n)}(\sum_{k=-\infty}^{j-1} Q_k)B_n(s)Y_{+,n}^{(j)}(s)\,ds,\ t\geq 0,\ j\in\mathbb{Z}_-;$$

$$\begin{aligned}Y_{-,n}^{(j)}(t) = T_-(-t)Q_j\mathcal{Q}_-^{(n)} + \int_{-\infty}^t \Big[T_+(t-s)\mathcal{Q}_+^{(n)} \\ + T_-(s-t)\mathcal{Q}_-^{(n)}(\sum_{k=1}^{j} Q_k)\Big]B_n(s)Y_{-,n}^{(j)}(s)\,ds \qquad (4.2)\\ - \int_t^0 T_-(s-t)\mathcal{Q}_-^{(n)}(\sum_{k=j+1}^{\infty} Q_k)B_n(s)Y_{-,n}^{(j)}(s)\,ds,\ t\leq 0,\ j\in\mathbb{Z}_-.\end{aligned}$$

We remark that for each n only finitely many of the solutions $Y_{\pm,n}^{(j)}$ of (4.1)-(4.2) are nonzero since $Q_j\mathcal{Q}^{(n)} = 0$ for $n < |j|$ and $Q_j\mathcal{Q}^{(n)} = Q_j$ for $n\geq |j|$.

Define $Z_{\pm,n}^{(j)} : \mathbb{R}_+ \to \mathcal{B}_2(\mathcal{X})$ by

$$Z_{\pm,n}^{(j)}(t) = e^{-\varkappa_j t}Y_{\pm,n}^{(j)}(t) \quad j\in\mathbb{Z}_\mp,\ n\in\mathbb{Z}_+. \tag{4.3}$$

Next, we recall the notation $\mathcal{C}_p^\pm = C_{\rm b}(\mathbb{R}_\pm,\mathcal{B}_p(\mathcal{X}))$, $p = 1,2,\dots\infty$ and $\mathcal{C}_{\rm b}^\pm = C_{\rm b}(\mathbb{R}_\pm,\mathcal{B}(\mathcal{X}))$. In addition, we define the operators $F_{j,n}^\pm$ by replacing the operator $B(t)$ in the definition (3.3) of $F_j^\pm$ by the operator $B_n(t) = \varphi_n^2(t)\mathcal{Q}^{(n)}B(t)\mathcal{Q}^{(n)}$. Similarly to (3.13), assuming Hypothesis 3.2 we can prove that

$$\|F_j^-\| \leq \int_{\mathbb{R}_-}\|B(t)\|_2\,dt < 1, \quad \|F_{j,n}^\pm\| \leq \int_{\mathbb{R}_\pm}\|B_n(t)\|_2\,dt \leq \int_{\mathbb{R}_\pm}\|B(t)\|_2\,dt < 1 \tag{4.4}$$

for all $n\geq 1$, where the norms of F_j^-, $F_{j,n}^\pm$ are the operator norms in the respective spaces of bounded and continuous functions.

We remark that if X_1 and X_2 are two Banach spaces, $X_2\hookrightarrow X_1$ and $F : X_1\to X_2$ is a linear operator with $\|F\| < 1$ then $(I_{X_k} - F)^{-1}$ is boundedly invertible from X_k to itself, for $k = 1,2$. Since $\mathcal{C}_2^\pm = \operatorname{im}(F_j^\pm) \subseteq \operatorname{dom}(F_j^\pm) = \mathcal{C}_{\rm b}^\pm$ and $\mathcal{C}_2^\pm = \operatorname{im}(F_{j,n}^\pm) \subseteq \operatorname{dom}(F_{j,n}^\pm) = \mathcal{C}_{\rm b}^\pm$ it follows from (4.4) that $I_{\mathcal{C}_p^\pm} - F_j^\pm$ and $I_{\mathcal{C}_p^\pm} - F_{j,n}^\pm$ are boundedly invertible from $\mathcal{C}_p^\pm$ to itself, where $p = 2,\mathrm{b}$. Moreover, from (3.11), (3.12), (4.1) and (4.2), respectively, we infer

$$\begin{aligned}Z_\pm^{(j)} &= (I_{\mathcal{C}_p^\pm} - F_j^\pm)^{-1}(e^{-\varkappa_j\cdot}T_\pm(\pm\cdot)Q_j)\\ Z_{\pm,n}^{(j)} &= (I_{\mathcal{C}_p^\pm} - F_{j,n}^\pm)^{-1},(e^{-\varkappa_j\cdot}T_\pm(\pm\cdot)\mathcal{Q}_\pm^{(n)}Q_j).\end{aligned} \tag{4.5}$$

In the next lemma we prove some basic approximation results for the generalized operator valued Jost solutions.

Lemma 4.1. *Assume Hypothesis 3.2. Then the following assertions hold true:*
(i) $\int_{\mathbb{R}}\|B_n(s) - B(s)\|_2\,ds \to 0$ *as* $n\to\infty$;
(ii) $\sup_{j\in\mathbb{Z}_\mp}\|(I_{\mathcal{C}_p^\pm} - F_{j,n}^\pm)^{-1} - (I_{\mathcal{C}_p^\pm} - F_j^\pm)^{-1}\|_{\mathcal{B}(\mathcal{C}_p^\pm)} \to 0$ *as* $n\to\infty$ *for* $p = 2,\mathrm{b}$;
(iii) $Z_{\pm,n}^{(j)} \to Z_\pm^{(j)}$ *in* $\mathcal{C}_2^\pm$ *as* $n\to\infty$;
(iv) $Z_{\pm,n}^{(j)} \to Z_\pm^{(j)}$ *in* $\mathcal{C}_{\rm b}^\pm$ *as* $n\to\infty$;
(v) There exists a constant $c > 0$ *independent of* $j\in\mathbb{Z}_\mp$ *such that*

$$\|Z_{\pm,n}^{(j)}\|_{\mathcal{C}_{\rm b}^\pm} \leq c \quad \text{for all} \quad n\in\mathbb{Z}_+,\ j\in\mathbb{Z}_\mp.$$

Proof. (i) Since $\varphi_n(s) \to 1$ as $n \to \infty$, $0 \le \varphi_n(s) \le 1$, $\mathcal{Q}^{(n)} \to I$ strongly as $n \to \infty$ and $B(s) \in \mathcal{B}_2(\mathcal{X})$ for all $s \in \mathbb{R}$, from [36, Ex. 3, Pg. 45] we deduce that $\varphi_n(s)\mathcal{Q}^{(n)}B(s) \to B(s)$ in $\mathcal{B}_2(\mathcal{X})$ as $n \to \infty$, and thus, $\varphi_n^2(s)B(s)^*\mathcal{Q}^{(n)} \to B(s)^*$ in $\mathcal{B}_2(\mathcal{X})$ as $n \to \infty$ for all $s \in \mathbb{R}$. Using again [36, Ex. 3, Pg. 45], and since $\|\mathcal{Q}^{(n)}\| \le 1$, one infers $B_n(s)^* = \varphi_n^2(s)\mathcal{Q}^{(n)}B(s)^*\mathcal{Q}^{(n)} \to B(s)^*$ in $\mathcal{B}_2(\mathcal{X})$ as $n \to \infty$, and hence, $B_n(s) \to B(s)$ in $\mathcal{B}_2(\mathcal{X})$ as $n \to \infty$. Moreover, $\|B_n(s) - B(s)\|_2 \le 2\|B(s)\|_2$ for all $s \in \mathbb{R}$ and $n \in \mathbb{Z}_+$, and so, using Lebesgue's Dominated Convergence Theorem we conclude $\int_{\mathbb{R}} \|B_n(s) - B(s)\|_2\,ds \to 0$ as $n \to \infty$.

(ii) Let $p = 2$ or $p = \mathrm{b}$. Using (3.6) we estimate

$$\|F_{j,n}^{\pm} - F_j^{\pm}\|_{\mathcal{B}(\mathcal{C}_p^{\pm})} \le \int_{\mathbb{R}_\pm} \|B_n(s) - B(s)\|_2\,ds \le \int_{\mathbb{R}} \|B_n(s) - B(s)\|_2\,ds$$

for all $n \in \mathbb{Z}_+$ and $j \in \mathbb{Z}_-$. Using estimates (4.4) we infer that there is a constant c independent of j such that

$$\begin{aligned}
&\|(I_{\mathcal{C}_p^{\pm}} - F_{j,n}^{\pm})^{-1} - (I_{\mathcal{C}_p^{\pm}} - F_j^{\pm})^{-1}\|_{\mathcal{B}(\mathcal{C}_p^{\pm})} = \\
&\quad = \|(I_{\mathcal{C}_p^{\pm}} - F_{j,n}^{\pm})^{-1}(F_{j,n}^{\pm} - F_j^{\pm})(I_{\mathcal{C}_p^{\pm}} - F_j^{\pm})^{-1}\|_{\mathcal{B}(\mathcal{C}_p^{\pm})} \\
&\quad \le \|(I_{\mathcal{C}_p^{\pm}} - F_{j,n}^{\pm})^{-1}\|_{\mathcal{B}(\mathcal{C}_p^{\pm})}\|F_{j,n}^{\pm} - F_j^{\pm}\|_{\mathcal{B}(\mathcal{C}_p^{\pm})}\|(I_{\mathcal{C}_p^{\pm}} - F_j^{\pm})^{-1}\|_{\mathcal{B}(\mathcal{C}_p^{\pm})} \\
&\quad \le c\|F_{j,n}^{\pm} - F_j^{\pm}\|_{\mathcal{B}(\mathcal{C}_p^{\pm})} \le \int_{\mathbb{R}} \|B_n(s) - B(s)\|_2\,ds
\end{aligned}$$

for all $n \in \mathbb{Z}_+$ and $j \in \mathbb{Z}_-$, which proves (ii).

(iii) Let $U_{n,j}^{\pm}, U_j^{\pm} : \mathbb{R}_\pm \to \mathcal{B}_2(\mathcal{X})$ be defined by $U_{n,j}^{\pm}(t) = e^{-\varkappa_j t}T_\pm(\pm t)\mathcal{Q}_\pm^{(n)}Q_j$ and $U_j^{\pm}(t) = e^{-\varkappa_j t}T_\pm(\pm t)Q_j$. Using (2.15) we estimate

$$\begin{aligned}
\|U_{n,j}^{\pm}(t) - U_j^{\pm}(t)\|_2 &= \|e^{-\varkappa_j t}T_\pm(\pm t)Q_j(Q_j\mathcal{Q}^{(n)} - Q_j)\|_2 \\
&\le e^{-\varkappa_j t}\|T_\pm(\pm t)Q_j\|\|Q_j\mathcal{Q}^{(n)} - Q_j)\|_2 = \|Q_j\mathcal{Q}^{(n)} - Q_j)\|_2
\end{aligned}$$

for all $t \in \mathbb{R}_\pm$, proving $U_{n,j}^{\pm} \to U_j^{\pm}$ in $\mathcal{C}_2^{\pm}$ as $n \to \infty$. Using assertion (ii) in the lemma we conclude

$$Z_{\pm,n}^{(j)} = (I_{\mathcal{C}_2^{\pm}} - F_{j,n}^{\pm})^{-1}U_{n,j}^{\pm} \to (I_{\mathcal{C}_2^{\pm}} - F_j^{\pm})^{-1}U_j^{\pm} = Z_{\pm}^{(j)} \quad \text{in} \quad \mathcal{C}_2^{\pm} \quad \text{as} \quad n \to \infty.$$

(iv) Follows immediately from (iii) and the definition of the spaces $\mathcal{C}_p^{\pm}$, $p = 2, \infty$.

(v) By (4.4), there is a constant $c > 0$, depending only on $\int_{\mathbb{R}_\pm} \|B(t)\|_2\,dt < 1$ (for example we can take $c = (1 - \int_{\mathbb{R}_\pm} \|B(t)\|_2\,dt)^{-1}$), such that

$$\|(I_{\mathcal{C}_{\mathrm{b}}^{\pm}} - F_{j,n}^{\pm})^{-1}\|_{\mathcal{B}(\mathcal{C}_{\mathrm{b}}^{\pm})} \le c \quad \text{for all} \quad n \in \mathbb{Z}_+,\ j \in \mathbb{Z}_\mp.$$

Moreover, we have that

$$\|U_{n,j}^{\pm}(t)\| = \|e^{-\varkappa_j t}T_\pm(\pm t)Q_j\mathcal{Q}^{(n)}\| \le e^{-\varkappa_j t}\|T_\pm(\pm t)Q_j\|\|\mathcal{Q}^{(n)}\| = 1.$$

Assertion (v) follows from the last two estimates and (4.5) . □

To define the determinants E_n, $n \in \mathbb{Z}_+$ we recall that for each $n \in \mathbb{Z}_+$ only finitely many of the solutions $Y_{\pm,n}^{(j)}$ of (4.1)-(4.2) are nonzero since

$$Q_j\mathcal{Q}^{(n)} = 0 \quad \text{for} \quad n < |j| \quad \text{and} \quad Q_j\mathcal{Q}^{(n)} = Q_j \quad \text{for} \quad n \ge |j|. \tag{4.6}$$

Thus, for each $n \in \mathbb{Z}_+$, the sum

$$\mathcal{Y}_{\pm,n} := \sum_{j \in \mathbb{Z}_\mp} Y_{\pm,n}^{(j)}(0), \tag{4.7}$$

has only finitely many non-zero terms and we can define

$$E_n := \det_{2,\mathrm{im}\,\mathcal{Q}^{(n)}}(\mathcal{Y}_{+,n} + \mathcal{Y}_{-,n}). \tag{4.8}$$

To prove that $E_n \to E$, for E from (3.29), we will approximate the linear operators $\mathcal{Y}_\pm - \mathcal{Q}_\pm$ in the $\mathcal{B}_2(\mathcal{X})$–norm. Recalling (3.16) and (3.17) let us define the linear operators $\tilde{Y}^{(j)}_{\pm,n}(0) : \mathcal{X} \to \mathcal{X}$, $j \in \mathbb{Z}_\mp$, $n \in \mathbb{Z}_+$ as follows:

$$\tilde{Y}^{(j)}_{+,n}(0) := Q_j - \int_0^\infty \Big[T_+(-s)(\sum_{k=j}^{-1} Q_k) + T_-(s)\mathcal{Q}_-\Big] B_n(s) Y^{(j)}_{+,n}(s)\, ds; \tag{4.9}$$

$$\tilde{Y}^{(j)}_{-,n}(0) := Q_j + \int_{-\infty}^0 \Big[T_+(-s)\mathcal{Q}_+ + T_-(s)(\sum_{k=1}^{j} Q_k)\Big] B_n(s) Y^{(j)}_{-,n}(s)\, ds. \tag{4.10}$$

Factoring $\mathcal{Q}^{(n)}$ in the integral equations (4.1)-(4.2) we remark that

$$Y^{(j)}_{\pm,n}(0) = \mathcal{Q}^{(n)} \tilde{Y}^{(j)}_{\pm,n}(0) \quad \text{for all} \quad j \in \mathbb{Z}_\mp,\ n \in \mathbb{Z}_+. \tag{4.11}$$

By (4.6), we also have:

$$\tilde{Y}^{(j)}_{\pm,n}(0) = Y^{(j)}_{\pm,n}(0) \quad \text{for} \quad n \geq |j| \quad \text{and} \quad \tilde{Y}^{(j)}_{\pm,n}(0) = Q_j + Y^{(j)}_{\pm,n}(0) \quad \text{for} \quad n < |j|. \tag{4.12}$$

In the next lemma we show that the series $\sum_{j \in \mathbb{Z}_\mp} \tilde{Y}^{(j)}_{\pm,n}(0)$ are strongly convergent.

Lemma 4.2. *Assume Hypotheses 2.2 and 3.2. Then:*

(i) *For each $n \in \mathbb{Z}_+$, the following series are strongly convergent*

$$\tilde{\mathcal{Y}}_{\pm,n} := \sum_{j \in \mathbb{Z}_\mp} \tilde{Y}^{(j)}_{\pm,n}(0);$$

(ii) *If $\mathbf{Z}_{\pm,n} = \{Z^{(j)}_{\pm,n}\}_{j \in \mathbb{Z}_\mp}$ are defined in* (4.3) *and $L^\pm_0$ and $L^\pm_S$ are defined in* (3.25)*and* (3.26) *then*

$$\tilde{\mathcal{Y}}_{\pm,n} = \mathcal{Q}_\pm \mp L^\pm_0(B_n, \mathbf{Z}_{\pm,n}) \mp L^\pm_S(B_n, \mathbf{Z}_{\pm,n});$$

(iii) *For each $n \in \mathbb{Z}_+$, we have that $\mathcal{Y}_{\pm,n} = \mathcal{Q}^{(n)} \tilde{\mathcal{Y}}_{\pm,n}$ and $\tilde{\mathcal{Y}}_{\pm,n} = \mathcal{Q}_\pm - \mathcal{Q}^{(n)}_\pm + \mathcal{Y}_{\pm,n}$.*

Proof. The proof of (i) and (ii) is similar to the proof of Theorem 3.6. From the definition of $\tilde{Y}^{(j)}_{\pm,n}(0)$ in (4.9)–(4.10) we have that

$$\tilde{Y}^{(j)}_{+,n}(0) = Q_j - L^+_{jj}(B_n, \mathbf{Z}_{+,n}) - \sum_{k>j} L^+_{kj}(B_n, \mathbf{Z}_{+,n}) \quad \text{for all} \quad j \in \mathbb{Z}_-,\ n \in \mathbb{Z}_+;$$

$$\tilde{Y}^{(j)}_{-,n}(0) = Q_j + L^-_{jj}(B_n, \mathbf{Z}_{-,n}) + \sum_{k<j} L^-_{kj}(B_n, \mathbf{Z}_{-,n}) \quad \text{for all} \quad j \in \mathbb{Z}_+,\ n \in \mathbb{Z}_+.$$

Since $\mathcal{Q}_\pm = \sum_{j \in \mathbb{Z}_\mp} Q_j$ strongly, (i) and (ii) follow from Lemma 3.4 (iii) and Lemma 3.5 (iii), while (iii) follows from (4.11) and (4.12). □

In the next lemma we show the main $\mathcal{B}_2(\mathcal{X})$-convergence result of this section.

Lemma 4.3. *Assume Hypotheses 2.2 and 3.2. Then*

$$\tilde{\mathcal{Y}}_{\pm,n} - \mathcal{Y}_\pm \to 0 \quad \text{in} \quad \mathcal{B}_2(\mathcal{X}) \quad \text{as} \quad n \to \infty.$$

Proof. From Theorem 3.6(ii) and Lemma 4.2(ii) it follows that to prove the lemma it is enough to show that

$$L_0^{\pm}(B_n,\mathbf{Z}_{\pm,n})\to L_0^{\pm}(B,\mathbf{Z}_{\pm})\quad\text{and}\quad L_S^{\pm}(B_n,\mathbf{Z}_{\pm,n})\to L_S^{\pm}(B,\mathbf{Z}_{\pm}) \tag{4.13}$$

in $\mathcal{B}_2(\mathcal{X})$ as $n\to\infty$. First, we note that $L_0^{\pm}(\cdot,\cdot)$ is a bilinear map and thus

$$L_0^{\pm}(B_n,\mathbf{Z}_{\pm,n})-L_0^{\pm}(B,\mathbf{Z}_{\pm})=L_0^{\pm}(B_n-B,\mathbf{Z}_{\pm,n})+L_0^{\pm}(B,\mathbf{Z}_{\pm,n}-\mathbf{Z}_{\pm}). \tag{4.14}$$

Using the estimates from Lemma 4.1(v) and Lemma 3.4(ii) we obtain

$$\begin{aligned}\|L_0^{\pm}(B_n-B,\mathbf{Z}_{\pm,n})\|_2&\le\|\mathbf{Z}_{\pm,n}\|_\infty\bigg(\int_{\mathbb{R}_\pm}\|B_n(s)-B(s)\|\,ds\bigg)\\&=\sup_{j\in\mathbb{Z}_\mp}\|Z_{\pm,n}^{(j)}\|_{\mathcal{C}_{\mathrm{b}}^{\pm}}\bigg(\int_{\mathbb{R}_\pm}\|B_n(s)-B(s)\|\,ds\bigg)\\&\le c\int_{\mathbb{R}}\|B_n(s)-B(s)\|\,ds.\end{aligned} \tag{4.15}$$

From this estimate and Lemma 4.1(i) we obtain

$$L_0^{\pm}(B_n-B,\mathbf{Z}_{\pm,n})\to 0\quad\text{in}\quad\mathcal{B}_2(\mathcal{X})\quad\text{as}\quad n\to\infty. \tag{4.16}$$

Using Lemma 3.4(ii) again, we estimate

$$\begin{aligned}\|L_0^{\pm}(B,\mathbf{Z}_{\pm,n}-\mathbf{Z}_{\pm})\|_2^2&\le\bigg(\int_{\mathbb{R}_\pm}\|B(s)\|_2\,ds\bigg)\sum_{j\in\mathbb{Z}_\mp}\sum_{m\in N_j}\Big[\|Z_{\pm,n}^{(j)}-Z_{\pm}^{(j)}\|_{\mathcal{C}_{\mathrm{b}}^{\pm}}^2h_m\Big]\\&=\bigg(\int_{\mathbb{R}_\pm}\|B(s)\|_2\,ds\bigg)\sum_{j\in\mathbb{Z}_\mp}\Big[\|Z_{\pm,n}^{(j)}-Z_{\pm}^{(j)}\|_{\mathcal{C}_{\mathrm{b}}^{\pm}}^2\big(\sum_{m\in N_j}h_m\big)\Big],\end{aligned}$$

where we temporarily denote $h_m=\int_{\mathbb{R}_\pm}\|\tilde{B}^*(s)v_m\|^2\,ds$ and $\tilde{B}(\cdot)$ is defined in (3.23). From Lemma 4.1(iv) we have that

$$\|Z_{\pm,n}^{(j)}-Z_{\pm}^{(j)}\|_{\mathcal{C}_{\mathrm{b}}^{\pm}}\to 0\quad\text{as}\quad n\to\infty\quad\text{for all}\quad j\in\mathbb{Z}_\mp. \tag{4.17}$$

Moreover, from Lemma 3.3(ii) and Lemma 4.1(v) we have that

$$\|Z_{\pm,n}^{(j)}-Z_{\pm}^{(j)}\|_{\mathcal{C}_{\mathrm{b}}^{\pm}}\le 2c\quad\text{for all}\quad j\in\mathbb{Z}_\mp,\ n\in\mathbb{Z}_+. \tag{4.18}$$

In addition,

$$\begin{aligned}\sum_{j\in\mathbb{Z}_\mp}\bigg(\sum_{m\in N_j}h_m\bigg)&\le\sum_{m\in\mathbb{Z}\setminus\{0\}}\int_{\mathbb{R}_\pm}\|\tilde{B}^*(s)v_m\|^2\,ds=\int_{\mathbb{R}_\pm}\sum_{m\in\mathbb{Z}\setminus\{0\}}\|\tilde{B}^*(s)v_m\|^2\,ds\\&=\int_{\mathbb{R}_\pm}\|\tilde{B}^*(s)\|_2^2\,ds=\int_{\mathbb{R}_\pm}\|B(s)\|_2\,ds<\infty.\end{aligned} \tag{4.19}$$

From (4.17)–(4.19) and Lebesgue's Dominated Convergence Theorem we conclude that

$$L_0^{\pm}(B,\mathbf{Z}_{\pm,n}-\mathbf{Z}_{\pm})\to 0\quad\text{in}\quad\mathcal{B}_2(\mathcal{X})\quad\text{as}\quad n\to\infty. \tag{4.20}$$

Finally, from (4.14), (4.16) and (4.20) we infer that

$$L_0^{\pm}(B_n,\mathbf{Z}_{\pm,n})\to L_0^{\pm}(B,\mathbf{Z}_{\pm})\quad\text{in}\quad\mathcal{B}_2(\mathcal{X})\quad\text{as}\quad n\to\infty,$$

proving the first part of (4.13). To prove the second assertion in (4.13), we remark that $L_S^{\pm}(\cdot,\cdot)$ is also a bilinear map, which implies

$$L_S^{\pm}(B_n,\mathbf{Z}_{\pm,n})-L_S^{\pm}(B,\mathbf{Z}_{\pm})=L_S^{\pm}(B_n-B,\mathbf{Z}_{\pm,n})+L_S^{\pm}(B,\mathbf{Z}_{\pm,n}-\mathbf{Z}_{\pm}). \tag{4.21}$$

Next, we will prove that

$$L^{\pm}_{kj}(B_n - B, \mathbf{Z}_{\pm,n}) \to 0 \quad \text{and} \quad L^{\pm}_{kj}(B, \mathbf{Z}_{\pm,n} - \mathbf{Z}_{\pm}) \to 0 \tag{4.22}$$

in $\mathcal{B}_2(\mathcal{X})$ as $n \to \infty$ for all $(k,j) \in \mathbb{Z}_{2,\pm}$. Using the first estimate in (3.22) and Lemma 4.1(v) we see that

$$\begin{aligned}\|L^{\pm}_{kj}(B_n - B, \mathbf{Z}_{\pm,n})\|_2 &\le \|Z^{(j)}_{\pm,n}\|_{\mathcal{C}^{\pm}_{\mathrm{b}}} \int_{\mathbb{R}_\pm} e^{(\varkappa_j - \varkappa_k)s} \|B_n(s) - B(s)\|_2\, ds \\ &\le c \int_{\mathbb{R}_\pm} \|B_n(s) - B(s)\|_2\, ds \le c \int_{\mathbb{R}} \|B_n(s) - B(s)\|_2\, ds\end{aligned}$$

for all $(k,j) \in \mathbb{Z}_{2,\pm}$, $n \in \mathbb{Z}_+$. From Lemma 4.1(i) it follows that

$$L^{\pm}_{kj}(B_n - B, \mathbf{Z}_{\pm,n}) \to 0 \quad \text{in} \quad \mathcal{B}_2(\mathcal{X}) \quad \text{as} \quad n \to \infty \tag{4.23}$$

for all $(k,j) \in \mathbb{Z}_{2,\pm}$. Using again the first estimate in (3.22), we obtain

$$\|L^{\pm}_{kj}(B, \mathbf{Z}_{\pm,n} - \mathbf{Z}_{\pm})\|_2 \le \|Z^{(j)}_{\pm,n} - Z^{(j)}_{\pm}\|_{\mathcal{C}^{\pm}_{\mathrm{b}}} \int_{\mathbb{R}_\pm} e^{(\varkappa_j - \varkappa_k)s} \|B(s)\|_2\, ds.$$

From this estimate and (4.17) we infer

$$L^{\pm}_{kj}(B, \mathbf{Z}_{\pm,n} - \mathbf{Z}_{\pm}) \to 0 \quad \text{in} \quad \mathcal{B}_2(\mathcal{X}) \quad \text{as} \quad n \to \infty \tag{4.24}$$

for all $(k,j) \in \mathbb{Z}_{2,\pm}$, which together with (4.23) proves (4.22).

Moreover, using the last estimate in (3.22), Lemma 4.1(v) and Lemma 3.3(ii), we obtain

$$\begin{aligned}\|L^{\pm}_{kj}(B_n - B, \mathbf{Z}_{\pm,n})\|_2 &\le \|\mathbf{Z}_{\pm,n}\|_\infty \Big(\sup_{s\in\mathbb{R}} \|B_n(s) - B(s)\|_2\Big) |\varkappa_k - \varkappa_j|^{-1} \\ &\le 2c \Big(\sup_{s\in\mathbb{R}} \|B(s)\|_2\Big) |\varkappa_k - \varkappa_j|^{-1};\end{aligned} \tag{4.25}$$

$$\begin{aligned}\|L^{\pm}_{kj}(B, \mathbf{Z}_{\pm,n} - \mathbf{Z}_{\pm})\|_2 &\le \|\mathbf{Z}_{\pm,n} - \mathbf{Z}_{\pm}\|_\infty \Big(\sup_{s\in\mathbb{R}} \|B(s)\|_2\Big) |\varkappa_k - \varkappa_j|^{-1} \\ &\le 2c \Big(\sup_{s\in\mathbb{R}} \|B(s)\|_2\Big) |\varkappa_k - \varkappa_j|^{-1}\end{aligned} \tag{4.26}$$

for all $(k,j) \in \mathbb{Z}_{2,\pm}$, $n \in \mathbb{Z}_{\pm}$. From (4.22), (4.25), (4.26), the estimate from Lemma 3.5(ii), Hypothesis 2.2 and Lebesgue's Dominated Convergence Theorem we conclude that

$$L^{\pm}_{S}(B_n - B, \mathbf{Z}_{\pm,n}) \to 0 \quad \text{and} \quad L^{\pm}_{S}(B, \mathbf{Z}_{\pm,n} - \mathbf{Z}_{\pm}) \to 0 \tag{4.27}$$

in $\mathcal{B}_2(\mathcal{X})$ as $n \to \infty$, proving the lemma. □

Next we prove the main result of this section saying that the Evans determinant can be approximated by a 2–modified Evans determinant on a finite–dimensional subspace.

Theorem 4.4. *Assume Hypotheses 2.2 and 3.2. Then*

$$E_n = \det{}_{2,\mathrm{im}\,\mathcal{Q}^{(n)}}(\mathcal{Y}_{+,n} + \mathcal{Y}_{-,n}) \to \det{}_{2,\mathcal{X}}(\mathcal{Y}_+ + \mathcal{Y}_-) = E \quad \textit{as} \quad n \to \infty.$$

Proof. From Lemma 4.2(ii) and (iii) we have that the operator

$$\tilde{\mathcal{Y}}_{+,n} + \tilde{\mathcal{Y}}_{-,n} - I = \mathcal{Y}_{+,n} + \mathcal{Y}_{-,n} - \mathcal{Q}^{(n)}$$

is Hilbert–Schmidt and thus $\det_{2,\mathcal{X}}(\tilde{\mathcal{Y}}_{+,n} + \tilde{\mathcal{Y}}_{-,n})$ is well–defined. Moreover,

$$E_n = \det_{2,\mathrm{im}\,\mathcal{Q}^{(n)}}(\mathcal{Y}_{+,n} + \mathcal{Y}_{-,n}) = \det_{2,\mathcal{X}}(\tilde{\mathcal{Y}}_{+,n} + \tilde{\mathcal{Y}}_{-,n}).$$

By Lemma 4.3, $\tilde{\mathcal{Y}}_{+,n} + \tilde{\mathcal{Y}}_{-,n} - \mathcal{Y}_{+} - \mathcal{Y}_{-} \to 0$ in $\mathcal{B}_2(\mathcal{X})$ as $n \to \infty$, which proves the result. □

5. The Evans determinant and the Birman–Schwinger operators

One of the main objectives of this paper is to relate the Evans determinant E defined in (3.29) to $\det_{2,L^2(\mathbb{R},X)}(I - \mathcal{K})$, where $\mathcal{K}$ is the Birman–Schwinger type operator defined in (2.19). We know that E can be approximated by E_n, see Theorem 4.4 and that $\det_{2,\mathcal{X}}(I - \mathcal{K})$ can be approximated by $\det_{2,L^2(\mathbb{R},X)}(I - \mathcal{K}_n)$, see Lemma 2.4(iii). Moreover, from [14, Thm. 7.8] we know that the finite–dimensional determinant $\det_{\mathrm{im}\,\mathcal{Q}^{(n)}}(\mathcal{Y}_{+,n} + \mathcal{Y}_{-,n})$ constructed by means of the generalized Jost solutions, see (4.1), (4.2) and (4.7), is related to the infinite–dimensional determinant $\det_{2,L^2(\mathbb{R},X)}(I - \mathcal{K}_n)$ as follows:

$$\det_{2,L^2(\mathbb{R},X)}(I - \mathcal{K}_n) = e^{\Theta^{(n)}} \det_{\mathrm{im}\,\mathcal{Q}^{(n)}} \Big(\mathcal{Y}_{+,n} + \mathcal{Y}_{-,n}\Big), \quad n \in \mathbb{Z}_+, \tag{5.1}$$

where

$$\Theta^{(n)} = \int_0^\infty \mathrm{tr}_{\mathrm{im}\,\mathcal{Q}^{(n)}} \left(\mathcal{Q}_+^{(n)} B_n(t)\right) dt - \int_{-\infty}^0 \mathrm{tr}_{\mathrm{im}\,\mathcal{Q}^{(n)}} \left(\mathcal{Q}_-^{(n)} B_n(t)\right) dt, \quad n \in \mathbb{Z}_+. \tag{5.2}$$

Using the fact that $\det(I+T) = \det_2(I+T)e^{\mathrm{tr}\,T}$ for any trace-class operator T, we conclude from (5.1) and (4.8) that

$$\det_{2,L^2(\mathbb{R},X)}(I - \mathcal{K}_n) = e^{\Theta^{(n)} - \Theta^{(n)}_{\mathrm{mod}}} E_n, \quad n \in \mathbb{Z}_+, \tag{5.3}$$

where the modification $\Theta^{(n)}_{\mathrm{mod}}$ of $\Theta^{(n)}$ from (5.2), is defined as follows:

$$\Theta^{(n)}_{\mathrm{mod}} = \mathrm{tr}_{\mathrm{im}\,\mathcal{Q}^{(n)}} \Big(I_{\mathrm{im}\,\mathcal{Q}^{(n)}} - \mathcal{Y}_{+,n} - \mathcal{Y}_{-,n}\Big), \quad n \in \mathbb{Z}_+. \tag{5.4}$$

Finally, to relate E and $\det_{2,L^2(\mathbb{R},X)}(I - \mathcal{K})$ we will pass to the limit as $n \to \infty$ in (5.3). In this section we will prove that the limit of $\Theta^{(n)} - \Theta^{(n)}_{\mathrm{mod}}$ as $n \to \infty$ indeed exists, and will compute it.

As in the previous section we assume Hypothesis 2.2 and Hypothesis 3.2. In particular, $B(t) \in \mathcal{B}_2(\mathcal{X})$ (but, in general, $B(t) \notin \mathcal{B}_1(\mathcal{X})$). Since $B_n(t)$ and $\mathcal{Y}_{\pm,n}$ are finite rank operators for all $t \in \mathbb{R}$, we can replace the traces in formulas (5.2) and (5.4) by traces in $\mathcal{X}$. Since the projection $\mathcal{Q}^{(n)} = \mathcal{Q}^{(n)}_+ + \mathcal{Q}^{(n)}_-$ can be identified with $I_{\mathrm{im}\,\mathcal{Q}^{(n)}}$ we see that

$$\Theta^{(n)} - \Theta^{(n)}_{\mathrm{mod}} = \Theta^{(n)}_+ + \Theta^{(n)}_-, \quad n \in \mathbb{Z}_+, \tag{5.5}$$

where

$$\Theta^{(n)}_+ = \mathrm{tr}_{\mathcal{X}} \left(\mathcal{Y}_{+,n} - \mathcal{Q}^{(n)}_+ + \int_0^\infty \mathcal{Q}^{(n)}_+ B_n(t)\, dt\right), \quad n \in \mathbb{Z}_+; \tag{5.6}$$

$$\Theta^{(n)}_- = \mathrm{tr}_{\mathcal{X}} \left(\mathcal{Y}_{-,n} - \mathcal{Q}^{(n)}_- - \int_{-\infty}^0 \mathcal{Q}^{(n)}_- B_n(t)\, dt\right), \quad n \in \mathbb{Z}_+. \tag{5.7}$$

Passing to the limit in formulas (5.6), (5.7) is not an easy task since in general $\int_{\mathbb{R}_\pm} B(t)\,dt$ is not a trace–class operator, and thus the limit of $\operatorname{tr}(\int_{\mathbb{R}_\pm} \mathcal{Q}_\pm^{(n)} B_n(t)dt)$ might be infinite. In the next lemma, we will modify formulas (5.6) and (5.7), to see that taking the limit is possible. In particular, we will extract the "diagonal" terms of the operators in (5.6) and (5.7) and show that this "diagonal" term converges in $\mathcal{B}_1$–norm to something manageable.

Lemma 5.1. *Assume Hypotheses 2.2 and 3.2. Then the following formulas hold:*

$$\Theta_\pm^{(n)} = \operatorname{tr}\left(\sum_{\mp j=1}^{n} \sum_{m\in N_j} \int_{\mathbb{R}_\pm}\int_{\mathbb{R}_\pm} B_n(t) W_{jm}^\pm(t,s) B_n(s) Z_{\pm,n}^{(j)}(s)\,ds\,dt\right), \tag{5.8}$$

where $Z_{\pm,n}^{(j)}$ are given in (4.3) and the functions $W_{jm}^\pm : \mathbb{R}_\pm^2 \to \mathcal{B}(\mathcal{X})$, $j \in \mathbb{Z}_\mp$, $m \in N_j$, are defined via $\Lambda_j^\pm$, $\Gamma_j^\pm$ from (3.4), (3.8) by

$$W_{jm}^\pm(t,s) = \begin{cases} e^{(\varkappa_j \pm a_m)t}\Lambda_j^\pm(s-t), & \text{if } \ 0 \le |t| \le |s|,\\ e^{(\varkappa_j \pm a_m)t}\Gamma_j^\pm(t-s), & \text{if } \ |t| > |s| \ge 0. \end{cases} \tag{5.9}$$

Proof. Using (4.1), (3.4) and (4.3) we obtain

$$\begin{aligned}
\Theta_+^{(n)} &= \sum_{m\in\mathbb{Z}\setminus\{0\}} \langle \mathcal{Y}_{+,n} v_m - \mathcal{Q}_+^{(n)} v_m + \int_0^\infty \mathcal{Q}_+^{(n)} B_n(t) v_m\,dt, v_m\rangle\\
&= \sum_{j=-n}^{-1}\sum_{m\in N_j} \langle Y_{+,n}^{(j)}(0) v_m - \mathcal{Q}_+^{(n)} Q_j v_m + \int_0^\infty \mathcal{Q}^{(n)} B_n(t) v_m\,dt, v_m\rangle\\
&= -\sum_{j=-n}^{-1}\sum_{m\in N_j} \int_0^\infty \Big(\langle \Lambda_j^+(t) B_n(t) Z_{+,n}^{(j)}(t) v_m, Q_j v_m\rangle - \langle B_n(t) v_m, v_m\rangle\Big)\,dt\\
&= -\sum_{j=-n}^{-1}\sum_{m\in N_j} \int_0^\infty \Big(\langle B_n(t) Z_{+,n}^{(j)}(t) v_m, (\Lambda_j^+(t))^* Q_j v_m\rangle - \langle B_n(t) v_m, v_m\rangle\Big)\,dt\\
&= -\sum_{j=-n}^{-1}\sum_{m\in N_j} \int_0^\infty \Big(\langle B_n(t) Z_{+,n}^{(j)}(t) v_m, e^{(\varkappa_j+\overline{a_m})t} v_m\rangle - \langle B_n(t) v_m, v_m\rangle\Big)\,dt\\
&= -\sum_{j=-n}^{-1}\sum_{m\in N_j} \int_0^\infty \langle B_n(t)\big(e^{(\varkappa_j+a_m)t} Z_{+,n}^{(j)}(t) - Q_j \mathcal{Q}_+^{(n)}\big) v_m, v_m\rangle\,dt.
\end{aligned} \tag{5.10}$$

From (4.1), (3.4) and (3.10), for all $n \in \mathbb{Z}_+$, $j \le -1$, $t \ge 0$, we infer

$$\begin{aligned}
e^{(\varkappa_j+a_m)t} Z_{+,n}^{(j)}(t) &= e^{a_m t} T_+(t) Q_j \mathcal{Q}_+^{(n)} - e^{(\varkappa_j+a_m)t}\int_t^\infty \Lambda_j^+(s-t) B_n(s) Z_{+,n}^{(j)}(s)\,ds\\
&\quad + e^{(\varkappa_j+a_m)t}\int_0^t \Gamma_j^+(t-s) B_n(s) Z_{+,n}^{(j)}(s)\,ds\\
&= Q_j \mathcal{Q}_+^{(n)} - e^{(\varkappa_j+a_m)t}\int_t^\infty \Lambda_j^+(s-t) B_n(s) Z_{+,n}^{(j)}(s)\,ds\\
&\quad + e^{(\varkappa_j+a_m)t}\int_0^t \Gamma_j^+(t-s) B_n(s) Z_{+,n}^{(j)}(s)\,ds.
\end{aligned}$$

Using this equation we can continue to simplify $\Theta_+^{(n)}$ in (5.10) as follows:

$$\begin{aligned}\Theta_+^{(n)} = &-\sum_{j=-n}^{-1}\sum_{m\in N_j}\int_0^\infty\int_0^t e^{(\varkappa_j+a_m)t}\langle B_n(t)\Gamma_j^+(t-s)B_n(s)Z_{+,n}^{(j)}(s)v_m,v_m\rangle\,ds\,dt\\ &+\sum_{j=-n}^{-1}\sum_{m\in N_j}\int_0^\infty\int_t^\infty e^{(\varkappa_j+a_m)t}\langle B_n(t)\Lambda_j^+(s-t)B_n(s)Z_{+,n}^{(j)}(s)v_m,v_m\rangle\,ds\,dt,\end{aligned}\tag{5.11}$$

proving the first part of (5.8) due to (5.9).

To prove the second part of (5.8), first, we recall that

$$\begin{aligned}Z_{-,n}^{(j)}(t) = e^{-\varkappa_j t}T_-(-t)Q_j\mathcal{Q}_-^{(n)} + \int_{-\infty}^t \Lambda_j^-(s-t)B_n(s)Z_{-,n}^{(j)}(s)\,ds\\ -\int_t^0\Gamma_j^-(t-s)B_n(s)Z_{-,n}^{(j)}(s)\,ds,\ t\le 0, j\in\mathbb{Z}_+, n\in\mathbb{Z}_+.\end{aligned}\tag{5.12}$$

Using (5.12), we can prove similarly to (5.11) that

$$\begin{aligned}\Theta_-^{(n)} = &\sum_{j=1}^{n}\sum_{m\in N_j}\int_{-\infty}^0\int_{-\infty}^t e^{(\varkappa_j-a_m)t}\langle B_n(t)\Lambda_j^-(s-t)B_n(s)Z_{-,n}^{(j)}(s)v_m,v_m\rangle\,ds\,dt\\ &-\sum_{j=1}^{n}\sum_{m\in N_j}\int_{-\infty}^0\int_t^0 e^{(\varkappa_j-a_m)t}\langle B_n(t)\Gamma_j^-(t-s)B_n(s)Z_{-,n}^{(j)}(s)v_m,v_m\rangle\,ds\,dt,\end{aligned}\tag{5.13}$$

which finishes the proof of lemma due to (5.9). □

In the remaining part of this section we will show how formulas (5.8) allow us to pass to the limit as $n\to\infty$ in (5.3). The crucial ingredient of the argument is that the RHS of (5.8) converges to a similarly looking expression but containing the operator $B(t)W_{jm}^+(t,s)B(s)Z_+^{(j)}(s)$ which is trace–class for all $(t,s)\in\mathbb{R}_+^2$.

From (3.6), (3.9) and the definition of $\varkappa_j$, $j\in\mathbb{Z}\setminus\{0\}$, we conclude that

$$\|W_{jm}^\pm(t,s)\|\le 1 \quad\text{for all}\quad (t,s)\in\mathbb{R}_\pm^2,\ j\in\mathbb{Z}_\mp,\ m\in N_j.\tag{5.14}$$

For any $j\in\mathbb{Z}\setminus\{0\}$ any two strongly continuous functions $B^{(1)},B^{(2)}:\mathbb{R}\to\mathcal{B}_2(\mathcal{X})$, satisfying the condition $\|B^{(1)}(\cdot)\|_2,\|B^{(2)}(\cdot)\|_2\in L^1(\mathbb{R})$, and any $\mathbf{Z}=\{Z^{(j)}\}_{j\in\mathbb{Z}_\mp}\in\ell^\infty(\mathbb{Z}_\mp,\mathcal{C}_{\rm b}^\pm)$, we define the diagonal operators $D_j^\pm(B^{(1)},B^{(2)},\mathbf{Z}):\mathcal{X}\to\mathcal{X}$, $j\in\mathbb{Z}_\mp$ by

$$D_j^\pm(B^{(1)},B^{(2)},\mathbf{Z})x = \sum_{m\in N_j}\langle x,v_m\rangle d_{jm}^\pm(B^{(1)},B^{(2)},\mathbf{Z})v_m,\tag{5.15}$$

where for any $j\in\mathbb{Z}_\mp$ and for $m\in N_j$ we introduced, for brevity, the notation $d_{jm}^\pm$ for the scalar multiplier on the main diagonal of $D_j^\pm(B^{(1)},B^{(2)},\mathbf{Z})$ as follows:

$$d_{jm}^\pm(B^{(1)},B^{(2)},\mathbf{Z}) = \int_{\mathbb{R}_\pm}\int_{\mathbb{R}_\pm}\langle B^{(1)}(t)W_{jm}^\pm(t,s)B^{(2)}(s)Z^{(j)}(s)v_m,v_m\rangle\,ds\,dt.\tag{5.16}$$

Remark 5.2. Using Lemma A.1 and (5.14), and since N_j is finite, we conclude that $D_j^{\pm}(B^{(1)}, B^{(2)}, \mathbf{Z}) \in \mathcal{B}_1(\mathcal{X})$ for all $j \in \mathbb{Z}_{\mp}$. Moreover, from (5.8) we infer that

$$\Theta_{\pm}^{(n)} = \operatorname{tr}\Big(\sum_{\mp j=1}^{n} D_j^{\pm}(B_n, B_n, \mathbf{Z}_{\pm,n})\Big) \quad \text{for all} \quad n \in \mathbb{Z}_+. \tag{5.17}$$

Next, we show that one can sum the operators $D_j^{\pm}(B_1, B_2, \mathbf{Z})$ over $j \in \mathbb{Z}_{\mp}$ to obtain diagonal operators, $D^{\pm}(B^{(1)}, B^{(2)}, \mathbf{Z})$. Later in this section we will show that the sum is a trace class operator and estimate its $\mathcal{B}_1$–norm. The trace of this diagonal trace class operator will be used to pass to the limit in (5.3).

Lemma 5.3. *Assume Hypothesis 2.2. For any $\mathbf{Z} \in \ell^{\infty}(\mathbb{Z}_{\mp}, \mathcal{C}_{\mathrm{b}}^{\pm})$ satisfying the condition $Z^{(j)}(s)Q_j = Z^{(j)}(s)$ for all $s \in \mathbb{R}_{\pm}, j \in \mathbb{Z}_{\mp}$ and any $B^{(1)}, B^{(2)} : \mathbb{R} \to \mathcal{B}_2(\mathcal{X})$, so that $\|B^{(1)}(\cdot)\|_2, \|B^{(2)}(\cdot)\|_2 \in L^1(\mathbb{R})$, we have:*

(i) *The linear operator $D^{\pm}(B^{(1)}, B^{(2)}, \mathbf{Z}) : \mathcal{X} \to \mathcal{X}$ defined by*

$$D^{\pm}(B^{(1)}, B^{(2)}, \mathbf{Z})x = \sum_{j \in \mathbb{Z}_{\mp}} D_j^{\pm}(B^{(1)}, B^{(2)}, \mathbf{Z})x \tag{5.18}$$

is well–defined and bounded;

(ii) *The absolute value of $D^{\pm}(B^{(1)}, B^{(2)}, \mathbf{Z})$ can be computed using the formula*

$$|D^{\pm}(B^{(1)}, B^{(2)}, \mathbf{Z})|x = \sum_{j \in \mathbb{Z}_{\mp}} \sum_{m \in N_j} \langle x, v_m\rangle |d_{jm}^{\pm}(B^{(1)}, B^{(2)}, \mathbf{Z})|, \tag{5.19}$$

where the scalar function $d_{jm}^{\pm}$ was defined in (5.16).

Proof. Through this proof we abbreviate $d_{jm}^{\pm} = d_{jm}^{\pm}(B^{(1)}, B^{(2)}, \mathbf{Z})$. To prove (i), it is enough to show that

$$\sum_{j \in \mathbb{Z}_{\mp}} \sum_{m \in N_j} \Big|\langle x, v_m\rangle d_{jm}^{\pm}\Big|^2 < \infty.$$

Using (5.14), we estimate

$$\begin{aligned} |d_{jm}^{\pm}| &\le \int_{\mathbb{R}_{\pm}} \int_{\mathbb{R}_{\pm}} \|B^{(1)}(t)\| \, \|B^{(2)}(s)\| \, \|D_j^{\pm}(t,s)\| \, \|Z^{(j)}(s)\| \, dt \, ds \\ &\le \|\mathbf{Z}\|_{\infty} \int_{\mathbb{R}_{\pm}} \int_{\mathbb{R}_{\pm}} \|B^{(1)}(t)\|_2 \, \|B^{(2)}(s)\|_2 \, dt \, ds, \end{aligned} \tag{5.20}$$

for all $j \in \mathbb{Z}_{\mp}, m \in N_j$. From the estimate above we obtain

$$\sum_{j \in \mathbb{Z}_{\mp}} \sum_{m \in N_j} \Big|\langle x, v_m\rangle d_{jm}^{\pm}\Big|^2 \le c \sum_{j \in \mathbb{Z}_{\mp}} \sum_{m \in N_j} |\langle x, v_m\rangle| = c\|\mathcal{Q}_{\pm}x\|^2 \le c\|x\|^2.$$

To prove (ii), we note that from the definition of the operators $D_j^{\pm}(B^{(1)}, B^{(2)}, \mathbf{Z})$, $j \in \mathbb{Z}_{\mp}$ in (5.15), it follows that $D^{\pm}(B^{(1)}, B^{(2)}, \mathbf{Z})$ is a diagonal operator, with the diagonal entries $d_{jm}^{\pm}$ for $m \in N_j$, $j \in \mathbb{Z}_{\mp}$. Thus, the operator $|D^{\pm}(B^{(1)}, B^{(2)}, \mathbf{Z})|$ is diagonal, with the diagonal entries $|d_{jm}^{\pm}|$ for $m \in N_j$, $j \in \mathbb{Z}_{\mp}$. □

In the next lemma we prove one of the main results of this section saying that the operators $D^{\pm}(B^{(1)}, B^{(2)}, \mathbf{Z})$ belong to $\mathcal{B}_1(\mathcal{X})$, and we estimate their norms. We define the sets

$$M_j := \{k \in \mathbb{Z} \setminus \{0\} : (k, j) \in \mathbb{Z}_{2,\pm}\}, \quad j \in \mathbb{Z}_{\mp}. \tag{5.21}$$

Lemma 5.4. *Assume Hypothesis 2.2. For any* $\mathbf{Z} \in \ell^\infty(\mathbb{Z}_\mp, \mathcal{C}_{\rm b}^\pm)$ *satisfying the condition* $Z^{(j)}(s)Q_j = Z^{(j)}(s)$ *for all* $s \in \mathbb{R}_\pm$, $j \in \mathbb{Z}_\mp$ *and any* $B^{(1)}, B^{(2)} : \mathbb{R} \to \mathcal{B}_2(\mathcal{X})$, *so that* $\|B^{(1)}(\cdot)\|_2, \|B^{(2)}(\cdot)\|_2 \in L^1(\mathbb{R})$ *we have:*

(i) *The operators* $D^\pm(B^{(1)}, B^{(2)}, \mathbf{Z})$ *belong to* $\mathcal{B}_1(\mathcal{X})$*;*

(ii) *The following estimates hold:*

$$\begin{aligned}\|D^\pm(B^{(1)}, B^{(2)}, \mathbf{Z})\|_1 &= \sum_{j\in\mathbb{Z}_\mp}\sum_{m\in N_j} |d^\pm_{jm}(B^{(1)}, B^{(2)}, \mathbf{Z})| \\ &\le c\|\mathbf{Z}\|_\infty \bigg(\int_{\mathbb{R}_\pm} \|B^{(1)}(t)\|_2\,dt\bigg)\bigg(\int_{\mathbb{R}_\pm} \|B^{(2)}(s)\|_2\,ds + \sup_{s\in\mathbb{R}} \|B^{(2)}(s)\|_2\bigg). \quad (5.22)\end{aligned}$$

Proof. By Lemma 5.3(ii) we know that for any $m \in N_j$

$$\langle |D^\pm(B^{(1)}, B^{(2)}, \mathbf{Z})| v_m, v_m\rangle = |d^\pm_{jm}(B^{(1)}, B^{(2)}, \mathbf{Z})|.$$

Thus, to prove the lemma it is enough to show that the sum

$$\sum_{j\in\mathbb{Z}_\mp}\sum_{m\in N_j} |d^\pm_{jm}(B^{(1)}, B^{(2)}, \mathbf{Z})| < \infty, \qquad (5.23)$$

and estimate the sum. This will be done in two steps. First, we will show

$$\begin{aligned}\sum_{m\in N_j} |d^\pm_{jm}(B^{(1)}, B^{(2)}, \mathbf{Z})| \le \|Z^{(j)}\|_{\mathcal{C}_{\rm b}^\pm}\bigg(\int_{\mathbb{R}_\pm}\int_{\mathbb{R}_\pm} \|B^{(1)}(t)^* Q_j\|_2\, \|B^{(2)}(s)^* Q_j\|_2\,dt\,ds \\ + \sup_{s\in\mathbb{R}} \|B^{(2)}(s)\|_2 \sum_{k\in M_j}\int_{\mathbb{R}_\pm} \|Q_k B^{(1)}(t)^* Q_j\|_2\,dt |\varkappa_k - \varkappa_j|^{-1}\bigg).\end{aligned} \qquad (5.24)$$

Next, we will use the last estimate to obtain (5.22).

In the proof of (5.24) we will abbreviate $d^\pm_{jm} = d^\pm_{jm}(B^{(1)}, B^{(2)}, \mathbf{Z})$. Since, in addition, $\sum_{k\in\mathbb{Z}\setminus\{0\}} Q_k = I$, *strongly*, we infer

$$\begin{aligned}d^\pm_{jm} &= \int_{\mathbb{R}_\pm}\int_{\mathbb{R}_\pm} \langle W^\pm_{jm}(t,s) B^{(2)}(s) Z^{(j)}(s) v_m, B^{(1)}(t)^* v_m\rangle\,ds\,dt \\ &= \int_{\mathbb{R}_\pm}\int_{\mathbb{R}_\pm} \langle W^\pm_{jm}(t,s) B^{(2)}(s) Z^{(j)}(s) v_m, \sum_{k\in\mathbb{Z}\setminus\{0\}} Q_k B^{(1)}(t)^* v_m\rangle\,ds\,dt \\ &= \int_{\mathbb{R}_\pm}\int_{\mathbb{R}_\pm} \sum_{k\in\mathbb{Z}\setminus\{0\}} \langle Q_k W^\pm_{jm}(t,s) B^{(2)}(s) Z^{(j)}(s) v_m, B^{(1)}(t)^* v_m\rangle\,ds\,dt. \quad (5.25)\end{aligned}$$

From the definitions of the integral kernels $W^\pm_{jm}$ in (5.9) we infer that there exists a scalar function $w^\pm_{kj}(t,s)$, $k \in \mathbb{Z}\setminus\{0\}$, $j \in \mathbb{Z}_\mp$, $m \in N_j$, such that

$$Q_k W^\pm_{jm}(t,s) = W^\pm_{jm}(t,s) Q_k = e^{(\varkappa_j \pm a_m)t} w^\pm_{kj}(t,s) Q_k. \qquad (5.26)$$

In addition, the function $w^\pm_{kj}$ has the following properties:

$$w^\pm_{kj}(t,s) = 0 \quad \text{if} \quad (k,j) \notin \mathbb{Z}_{2,\pm} \quad \text{and} \quad k \ne j; \qquad (5.27)$$

$$0 \le w^\pm_{kj}(t,s) \le 1 \quad \text{for all} \quad k \in \mathbb{Z}\setminus\{0\}, j \in \mathbb{Z}_\mp; \qquad (5.28)$$

$$\int_{\mathbb{R}_\pm} w^\pm_{kj}(t,s)\,ds = |\varkappa_k - \varkappa_j|^{-1} \quad \text{for all} \quad k \in \mathbb{Z}\setminus\{0\}, j \in \mathbb{Z}_\mp, k \ne j. \qquad (5.29)$$

Next, for $j\in\mathbb{Z}_{\mp}, m\in N_j$ we insert Q_j in (5.16) to define

$$\tilde d^{\pm}_{jm}=\int_{\mathbb{R}_\pm}\int_{\mathbb{R}_\pm}\langle Q_jW^{\pm}_{jm}(t,s)B^{(2)}(s)Z^{(j)}(s)v_m, B^{(1)}(t)^*v_m\rangle\,ds\,dt. \tag{5.30}$$

From (5.25), (5.27) and the definition of the sets M_j in (5.21) we conclude that

$$d^{\pm}_{jm}=\tilde d^{\pm}_{jm}+\sum_{k\in M_j}\int_{\mathbb{R}_\pm}\int_{\mathbb{R}_\pm}e^{(\varkappa_j\pm a_m)t}w^{\pm}_{kj}(t,s)\langle Q_kB^{(2)}(s)Z^{(j)}(s)v_m, B^{(1)}(t)^*v_m\rangle dsdt. \tag{5.31}$$

Using (5.26) and (5.14), we estimate:

$$\begin{aligned}\sum_{m\in N_j}|\tilde d^{\pm}_{jm}| &= \sum_{m\in N_j}\Big|\int_{\mathbb{R}_\pm}\int_{\mathbb{R}_\pm}\langle W^{\pm}_{jm}(t,s)Q_jB^{(2)}(s)Z^{(j)}(s)v_m, B^{(1)}(t)^*v_m\rangle\,ds\,dt\Big|\\
&\le \int_{\mathbb{R}_\pm}\int_{\mathbb{R}_\pm}\sum_{m\in N_j}\|Q_jB^{(2)}(s)Z^{(j)}(s)v_m\|\,\|B^{(1)}(t)^*v_m\|\,ds\,dt\\
&\le \int_{\mathbb{R}_\pm}\int_{\mathbb{R}_\pm}(\sum_{m\in N_j}\|Q_jB^{(2)}(s)Z^{(j)}(s)v_m\|^2)^{\frac12}(\sum_{m\in N_j}\|B^{(1)}(t)^*v_m\|^2)^{\frac12}\,ds\,dt\\
&= \int_{\mathbb{R}_\pm}\int_{\mathbb{R}_\pm}\|Q_jB^{(2)}(s)Z^{(j)}(s)\|_2\,\|B^{(1)}(t)^*Q_j\|_2\,ds\,dt\\
&= \int_{\mathbb{R}_\pm}\int_{\mathbb{R}_\pm}\|Z^{(j)}(s)^*B^{(2)}(s)^*Q_j\|_2\,\|B^{(1)}(t)^*Q_j\|_2\,ds\,dt\\
&\le \|Z^{(j)}\|_{\mathcal{C}_{\rm b}^{\pm}}\int_{\mathbb{R}_\pm}\int_{\mathbb{R}_\pm}\|B^{(1)}(t)^*Q_j\|_2\,\|B^{(2)}(s)^*Q_j\|_2\,ds\,dt.\end{aligned} \tag{5.32}$$

In addition, using (5.29), we estimate:

$$\begin{aligned}&\Big|\sum_{m\in N_j}\int_{\mathbb{R}_\pm}\int_{\mathbb{R}_\pm}e^{(\varkappa_j\pm a_m)t}w^{\pm}_{kj}(t,s)\langle Q_kB^{(2)}(s)Z^{(j)}(s)v_m, B^{(1)}(t)^*v_m\rangle\,ds\,dt\Big|\\
&\le \int_{\mathbb{R}_\pm}\int_{\mathbb{R}_\pm}w^{\pm}_{kj}(t,s)\sum_{m\in N_j}\|B^{(2)}(s)Z^{(j)}(s)v_m\|\,\|Q_kB^{(1)}(t)^*v_m\|\,ds\,dt\\
&\le \int_{\mathbb{R}_\pm}\int_{\mathbb{R}_\pm}w^{\pm}_{kj}(t,s)(\sum_{m\in N_j}\|B^{(2)}(s)Z^{(j)}(s)v_m\|^2)^{\frac12}(\sum_{m\in N_j}\|Q_kB^{(1)}(t)^*v_m\|^2)^{\frac12}\,ds\,dt\\
&\le \int_{\mathbb{R}_\pm}\int_{\mathbb{R}_\pm}w^{\pm}_{kj}(t,s)\|B^{(2)}(s)Z^{(j)}(s)\|_2\,\|Q_kB^{(1)}(t)^*Q_j\|_2\,ds\,dt\\
&\le \|Z^{(j)}\|_{\mathcal{C}_{\rm b}^{\pm}}\sup_{s\in\mathbb{R}}\|B^{(2)}(s)\|_2\int_{\mathbb{R}_\pm}\int_{\mathbb{R}_\pm}w^{\pm}_{kj}(t,s)\,\|Q_kB^{(1)}(t)^*Q_j\|_2\,ds\,dt\\
&= \|Z^{(j)}\|_{\mathcal{C}_{\rm b}^{\pm}}\sup_{s\in\mathbb{R}}\|B^{(2)}(s)\|_2\int_{\mathbb{R}_\pm}\Big(\|Q_kB^{(1)}(t)^*Q_j\|_2\int_{\mathbb{R}_\pm}w^{\pm}_{kj}(t,s)\;ds\Big)dt\\
&= \|Z^{(j)}\|_{\mathcal{C}_{\rm b}^{\pm}}\sup_{s\in\mathbb{R}}\|B^{(2)}(s)\|_2\,|\varkappa_k-\varkappa_j|^{-1}\int_{\mathbb{R}_\pm}\|Q_kB^{(1)}(t)^*Q_j\|_2\,dt.\end{aligned} \tag{5.33}$$

Estimate (5.24) easily follows from (5.31), and estimates (5.32) and (5.33) . We are now ready to finish the proof of (5.22).

It remains to show that

$$\sum_{j\in\mathbb{Z}_\mp}\int_{\mathbb{R}_\pm}\int_{\mathbb{R}_\pm}\|B^{(1)}(t)^*Q_j\|_2\,\|B^{(2)}(s)^*Q_j\|_2\,dt\,ds<\infty, \tag{5.34}$$

$$\sum_{j\in\mathbb{Z}_\mp}\sum_{k\in M_j}\int_{\mathbb{R}_\pm}\|Q_kB^{(1)}(t)^*Q_j\|_2\,dt|\varkappa_k-\varkappa_j|^{-1}<\infty, \tag{5.35}$$

and to estimate these sums. To prove (5.34) we estimate:

$$\begin{aligned}
&\sum_{j\in\mathbb{Z}_\mp}\int_{\mathbb{R}_\pm}\int_{\mathbb{R}_\pm}\|B^{(1)}(t)^*Q_j\|_2\,\|B^{(2)}(s)^*Q_j\|_2\,dt\,ds\\
&\qquad\le\int_{\mathbb{R}_\pm}\int_{\mathbb{R}_\pm}\big(\sum_{j\in\mathbb{Z}_\mp}\|B^{(1)}(t)^*Q_j\|_2^2\big)^{\frac12}\big(\sum_{j\in\mathbb{Z}_\mp}\|B^{(2)}(s)^*Q_j\|_2^2\big)^{\frac12}\,dt\,ds\\
&\qquad=\int_{\mathbb{R}_\pm}\int_{\mathbb{R}_\pm}\|B^{(1)}(t)^*\|_2\|B^{(2)}(s)^*\|_2\,dt\,ds\\
&\qquad=\int_{\mathbb{R}_\pm}\int_{\mathbb{R}_\pm}\|B^{(1)}(t)\|_2\|B^{(2)}(s)\|_2\,dt\,ds.
\end{aligned} \tag{5.36}$$

Finally, we estimate (5.35) as follows:

$$\begin{aligned}
&\sum_{j\in\mathbb{Z}_\mp}\sum_{k\in M_j}\int_{\mathbb{R}_\pm}\|Q_kB^{(1)}(t)^*Q_j\|_2\,dt|\varkappa_k-\varkappa_j|^{-1}\\
&\quad=\sum_{(k,j)\in\mathbb{Z}_{2,\pm}}\int_{\mathbb{R}_\pm}\|Q_kB^{(1)}(t)^*Q_j\|_2\,dt|\varkappa_k-\varkappa_j|^{-1}\\
&\quad=\int_{\mathbb{R}_\pm}\sum_{(k,j)\in\mathbb{Z}_{2,\pm}}\|Q_kB^{(1)}(t)^*Q_j\|_2|\varkappa_k-\varkappa_j|^{-1}\,dt\\
&\quad=\int_{\mathbb{R}_\pm}\Big(\sum_{(k,j)\in\mathbb{Z}_{2,\pm}}\|Q_kB^{(1)}(t)^*Q_j\|_2^2\Big)^{\frac12}\Big(\sum_{(k,j)\in\mathbb{Z}_{2,\pm}}(|\varkappa_k-\varkappa_j|^{-2}\Big)^{\frac12}\,dt\\
&\quad\le c\int_{\mathbb{R}_\pm}\Big(\sum_{k,j\in\mathbb{Z}\setminus\{0\}}\|Q_kB^{(1)}(t)^*Q_j\|_2^2\Big)^{\frac12}\,dt\\
&\quad=c\int_{\mathbb{R}_\pm}\Big(\sum_{k\in\mathbb{Z}\setminus\{0\}}\sum_{j\in\mathbb{Z}\setminus\{0\}}\sum_{m\in N_j}\|Q_kB^{(1)}(t)^*v_m\|^2\Big)^{\frac12}\,dt\\
&\quad=c\int_{\mathbb{R}_\pm}\Big(\sum_{j\in\mathbb{Z}\setminus\{0\}}\sum_{m\in N_j}\sum_{k\in\mathbb{Z}\setminus\{0\}}\|Q_kB^{(1)}(t)^*v_m\|^2\Big)^{\frac12}\,dt\\
&\quad=c\int_{\mathbb{R}_\pm}\Big(\sum_{j\in\mathbb{Z}\setminus\{0\}}\sum_{m\in N_j}\|B^{(1)}(t)^*v_m\|^2\Big)^{\frac12}\,dt\\
&\quad=c\int_{\mathbb{R}_\pm}\|B^{(1)}(t)\|_2\,dt.
\end{aligned} \tag{5.37}$$

This gives the desired estimate (5.35), finishing the proof of the lemma. □

The next theorem is the main technical result of this section. It proves the convergence of the sequences $\Theta_{\pm}^{(n)}$, $n\in\mathbb{Z}_+$, introduced in (5.6)–(5.7), to the trace of the diagonal operator $D^{\pm}(B,B,\mathbf{Z}_{\pm})$.

Theorem 5.5. *Assume Hypotheses 2.2 and 3.2 and let* $\mathbf{Z}_{\pm,n}=\{Z_{\pm,n}^{(j)}\}_{j\in\mathbb{Z}_{\mp}}$ *and* $\mathbf{Z}_{\pm}=\{Z_{\pm}^{(j)}\}_{j\in\mathbb{Z}_{\mp}}$ *defined in* (4.3) *and* (3.11)–(3.12)*, respectively. Then, for the operators* $D^{\pm}$ *defined in* (5.18)*, one has:*

(i) $D^{\pm}(B_n,B_n,\mathbf{Z}_{\pm,n})\to D^{\pm}(B,B,\mathbf{Z}_{\pm})$ *in* $\mathcal{B}_1(\mathcal{X})$ *as* $n\to\infty$*;*

(ii) $\Theta_{\pm}^{(n)}\to\operatorname{tr}(D^{\pm}(B,B,\mathbf{Z}_{\pm}))$ *as* $n\to\infty$.

Proof. (i) The operator $D^{\pm}(B_n,B_n,\mathbf{Z}_{\pm,n})-D^{\pm}(B,B,\mathbf{Z}_{\pm})$ is diagonal with the diagonal entries $d_{jm}^{\pm}(B_n,B_n,\mathbf{Z}_{\pm,n})-d_{jm}^{\pm}(B,B,\mathbf{Z}_{\pm})$, $m\in N_j$, $j\in\mathbb{Z}_{\mp}$. Thus, the operator $\big|D^{\pm}(B_n,B_n,\mathbf{Z}_{\pm,n})-D^{\pm}(B,B,\mathbf{Z}_{\pm})\big|$ is diagonal with the diagonal entries $|d_{jm}^{\pm}(B_n,B_n,\mathbf{Z}_{\pm,n})-d_{jm}^{\pm}(B,B,\mathbf{Z}_{\pm})|$, $m\in N_j$, $j\in\mathbb{Z}_{\mp}$. It follows that

$$\begin{aligned}&\|D^{\pm}(B_n,B_n,\mathbf{Z}_{\pm,n})-D^{\pm}(B,B,\mathbf{Z}_{\pm})\|_1\\&\qquad=\sum_{j\in\mathbb{Z}_{\mp}}\sum_{m\in N_j}|d_{jm}^{\pm}(B_n,B_n,\mathbf{Z}_{\pm,n})-d_{jm}^{\pm}(B,B,\mathbf{Z}_{\pm})|.\end{aligned}\tag{5.38}$$

To prove the convergence to 0 as $n\to\infty$ of the RHS of (5.38) we use Lebesgue's Dominated Convergence Theorem. Since $d_{jm}^{\pm}(\cdot,\cdot,\cdot)$ is a multilinear map, from estimate (5.20) and Lemma 4.1(i) and (iv) we conclude that

$$d_{jm}^{\pm}(B_n,B_n,\mathbf{Z}_{\pm,n})\to d_{jm}^{\pm}(B,B,\mathbf{Z}_{\pm})\quad\text{as}\quad n\to\infty\quad\text{for all}\quad j\in\mathbb{Z}_{\mp},m\in N_j.\tag{5.39}$$

Using (5.24) for $B^{(1)}=B^{(2)}=B_n(\cdot)=\mathcal{Q}^{(n)}B(\cdot)\mathcal{Q}^{(n)}$ we estimate:

$$\begin{aligned}&\sum_{m\in N_j}|d_{jm}^{\pm}(B_n,B_n,\mathbf{Z}_{\pm,n})|\\
&\leq\sum_{m\in N_j}\|Z_{\pm,n}^{(j)}\|_{\mathcal{C}_{\rm b}^{\pm}}\Big(\int_{\mathbb{R}_{\pm}}\int_{\mathbb{R}_{\pm}}\|B_n(t)^*Q_j\|_2\,\|B_n(s)^*Q_j\|_2\,dt\,ds\\
&\qquad+\sup_{s\in\mathbb{R}}\|B_n(s)\|_2\sum_{k\in M_j}\int_{\mathbb{R}_{\pm}}\|Q_kB_n(t)^*Q_j\|_2\,dt|\varkappa_k-\varkappa_j|^{-1}\Big)\\
&\leq c\sum_{m\in N_j}\Big(\int_{\mathbb{R}_{\pm}}\int_{\mathbb{R}_{\pm}}\|\mathcal{Q}^{(n)}B(t)^*\mathcal{Q}^{(n)}Q_j\|_2\,\|\mathcal{Q}^{(n)}B(s)^*\mathcal{Q}^{(n)}Q_j\|_2\,dt\,ds\\
&\qquad+\sum_{k\in M_j}\int_{\mathbb{R}_{\pm}}\|Q_k\mathcal{Q}^{(n)}B(t)^*\mathcal{Q}^{(n)}Q_j\|_2\,dt|\varkappa_k-\varkappa_j|^{-1}\Big)\\
&\leq c\sum_{m\in N_j}\Big(\int_{\mathbb{R}_{\pm}}\int_{\mathbb{R}_{\pm}}\|B(t)^*Q_j\|_2\,\|B(s)^*Q_j\|_2\,dt\,ds\\
&\qquad+\sum_{k\in M_j}\int_{\mathbb{R}_{\pm}}\|Q_kB(t)^*Q_j\|_2\,dt|\varkappa_k-\varkappa_j|^{-1}\Big)|.\end{aligned}\tag{5.40}$$

Moreover, denoting by p_j the RHS of (5.40), we infer

$$\sum_{m\in N_j}|d_{jm}^{\pm}(B_n,B_n,\mathbf{Z}_{\pm,n})-d_{jm}^{\pm}(B,B,\mathbf{Z}_{\pm})|\leq p_j+\sum_{m\in N_j}|d_{jm}^{\pm}(B,B,\mathbf{Z}_{\pm})|\tag{5.41}$$

Using (5.23), (5.34) and (5.35) for $B^{(1)} = B^{(2)} = B$ we have that

$$\sum_{j\in\mathbb{Z}_{\mp}} \big(p_j + \sum_{m\in N_j} |d^{\pm}_{jm}(B, B, \mathbf{Z}_{\pm})|\big) < \infty. \tag{5.42}$$

From Lebesgue's Dominated Convergence theorem, (5.39), (5.41) and (5.42) we obtain assertion (i) in the lemma.

(ii) By Remark 5.2 and the definition of the operators $D^{\pm}$ in Lemma 5.3(i) we infer that

$$\Theta^{(n)}_{\pm} = \operatorname{tr}(D^{\pm}(B_n, B_n, \mathbf{Z}_{\pm,n})) \tag{5.43}$$

for all $n \in \mathbb{Z}_+$. Using the convergence result in assertion (i), the lemma follows. □

In the remaining part of this section, we will show that the value of $\lim_{n\to\infty} \Theta^{(n)}_{\pm}$ computed in Theorem 5.5(ii) is related to the sum $\mathcal{Y}_+ + \mathcal{Y}_-$ of generalized operator valued Jost solutions at $t = 0$, as defined in Theorem 3.6(i). First, we define the *diagonal part* of a bounded linear operator T on $\mathcal{X}$ as follows:

$$\operatorname{diag}(T)x = \sum_{m\in\mathbb{Z}\setminus\{0\}} \langle x, v_m\rangle\langle Tv_m, v_m\rangle v_m, \quad x \in \mathcal{X}. \tag{5.44}$$

Remark 5.6. The following assertions hold true:

(i) $\operatorname{diag}(T) \in \mathcal{B}(\mathcal{X})$ for all $T \in \mathcal{B}(\mathcal{X})$;
(ii) $\operatorname{diag}(T) \in \mathcal{B}_1(\mathcal{X})$ and $\operatorname{tr}(\operatorname{diag}(T)) = \operatorname{tr}(T)$ for all $T \in \mathcal{B}_1(\mathcal{X})$.

Lemma 5.7. *Assume Hypotheses 2.2 and 3.2. If $\mathbf{Z}_{\pm} = \{Z^{(j)}_{\pm}\}_{j\in\mathbb{Z}_{\mp}}$ are as defined in* (3.11)–(3.12)*, then*

(i) $\operatorname{diag}\Big(\mathcal{Y}_{\pm} - \mathcal{Q}_{\pm} \pm \int_0^{\infty} \mathcal{Q}_{\pm}B(t)\,dt\Big) = D^{\pm}(B, B, \mathbf{Z}_{\pm}) \in \mathcal{B}_1(\mathcal{X})$*;*
(ii) $\Theta^{(n)}_{\pm} \to \operatorname{tr}\Big(\operatorname{diag}\big(\mathcal{Y}_{\pm} - \mathcal{Q}_{\pm} \pm \int_0^{\infty} \mathcal{Q}_{\pm}B(t)\,dt\big)\Big)$ *as $n \to \infty$.*

Proof. To prove (i), it is enough to show that

$$\Big\langle \Big(\mathcal{Y}_{\pm} - \mathcal{Q}_{\pm} \pm \int_{\mathbb{R}_{\pm}} \mathcal{Q}_{\pm}B(t)\,dt\Big) v_m, v_m\Big\rangle = \langle D^{\pm}(B, B, \mathbf{Z}_{\pm})v_m, v_m\rangle \tag{5.45}$$

for all $m \in \mathbb{Z} \setminus \{0\}$. The proof of (5.45) is similar to (5.10)–(5.11). Assertion (ii) follows from Theorem 5.5(ii) and (i). □

We conclude this section with one of the main results of this paper connecting the Evans determinant defined in (3.29) and $\det_{2,L^2(\mathbb{R},X)}(I - \mathcal{K})$, where $\mathcal{K}$ is the Birman–Schwinger type operator defined in (2.19).

Theorem 5.8. *Assume Hypotheses 2.2 and 3.2. The Evans determinant $E = \det_{2,\mathcal{X}}(\mathcal{Y}_+ + \mathcal{Y}_-)$ and the Birman–Schwinger operator $\mathcal{K}$ are related by the formula*

$$\det_{2,L^2(\mathbb{R},\mathcal{X})}(I - \mathcal{K}) = e^{\Xi} \det_{2,\mathcal{X}}(\mathcal{Y}_+ + \mathcal{Y}_-),$$

where we denote

$$\Xi = \operatorname{tr}\left[\operatorname{diag}\left(\mathcal{Y}_+ + \mathcal{Y}_- - I + \int_{\mathbb{R}_+} \mathcal{Q}_+B(t)\,dt - \int_{\mathbb{R}_-} \mathcal{Q}_-B(t)\,dt\right)\right].$$

Proof. This follows from (5.3)–(5.7), Lemma 2.4(iii), Theorem 4.4 and Lemma 5.7. □

6. The Evans function of second order differential operators

In many cases the spectral problem $\mathcal{L}U = \lambda U$ for a linear differential operator $\mathcal{L}$ can be recast as a first order ODE on a Hilbert space $\mathcal{X}$ of the form

$$u'(t) = [A(\lambda) + B(t)]u(t), \quad t \in \mathbb{R}, \tag{6.1}$$

where, for each λ, $A(\lambda)$ is a linear operator with discrete spectrum and B is an operator valued function with values in $\mathcal{B}(\mathcal{X})$. Under certain conditions, the Evans determinant for (6.1), defined in (3.29), is an analytic function in λ, whose zeros are the eigenvalues of the underlying differential operator $\mathcal{L}$.

In this section we construct the Evans function for a very general class of second order differential operators. To introduce these differential operators, we start by considering a separable Hilbert space $\mathcal{X}_0$ and two closed densly defined linear operators $A_0 : \mathrm{dom}(A_0) \subseteq \mathcal{X}_0 \to \mathcal{X}_0$ and $\Gamma : \mathrm{dom}(\Gamma) \subseteq \mathcal{X}_0 \to \mathcal{X}_0$ who share the set $\{e_n, n \in \mathbb{Z}_+\}$ of eigenvectors forming an orthonormal basis in $\mathcal{X}_0$.

Hypothesis 6.1. *The linear operators A_0 and Γ satisfy the following assumptions:*

(A_1) $A_0 e_n = \alpha_n e_n$, $\Gamma e_n = \gamma_n e_n$, *for some* $\alpha_n, \gamma_n \in \mathbb{C}$, *for all* $n \in \mathbb{Z}_+$*;*
(A_2) Γ *is boundedly invertible on* $\mathcal{X}_0$*;*
(A_3) $|\frac{\alpha_n}{\gamma_n}| = \mathcal{O}(n^\nu)$ *as* $n \to \infty$ *for some* $\nu > 0$ *and* $\alpha_n \gamma_n < 0$ *for all* $n \in \mathbb{Z}_+$.

As an immediate consequence of our assumptions in Hypothesis 6.1, we remark that the linear operators A_0 and Γ satisfy the following properties:

(i) $\sigma(A_0) = \{\alpha_n : n \in \mathbb{Z}_+\}$ is a discrete sets of semi–simple eigenvalues of finite multiplicity;
(ii) eigenvalues of Γ are allowed to have infinite multiplicity;
(iii) $|\alpha_n| \to \infty$ as $n \to \infty$.

A typical example of such operators is furnished by differential operators on $L^2(a, b)$, with $\Gamma = I$ or $\Gamma = \partial_\eta$, equipped with Dirichelet boundary conditions. For more sophisticated examples we refer to the next section.

The main result of this section will focus on the operator $\mathcal{L} : H^2(\mathbb{R}, \mathrm{dom}(A_0)) \to L^2(\mathbb{R}, \mathcal{X}_0)$ defined by

$$\mathcal{L} = \Gamma^{-1}\partial_t^2 + \Big(B_1(t)\Gamma^{-1} - c\Big)\partial_t + \Big(A_0 + B_0(t)\Big), \quad t \in \mathbb{R} \tag{6.2}$$

where $c \in \mathbb{R}$ and in addition we assume the following:

Hypothesis 6.2. *The operator valued functions B_0 and B_1 satisfy the following assumptions*

(B_0) $B_0 : \mathbb{R} \to \mathcal{B}_2(\mathrm{dom}(|A_0\Gamma^{-1}|^{1/2}), \mathcal{X}_0)$ *is bounded and strongly continuous;*
(B_1) $B_1 : \mathbb{R} \to \mathcal{B}_2(\mathcal{X}_0)$ *is bounded and strongly continuous.*

Next, we reduce the eigenvalue problem $\mathcal{L}U = \lambda U$ to a first order ODE system using the substitution $V = \Gamma^{-1}U'$:

$$\begin{cases} U'(t) = \Gamma V(t) \\ V'(t) = \lambda U(t) - \Big(B_1(t) - c\Gamma\Big)V(t) - \Big(A_0 + B_0(t)\Big)U(t) \end{cases} \tag{6.3}$$

Making the substitution $u(t) = (U(t), V(t))^{\mathrm{T}}$ we obtain the equation

$$u'(t) = \Big(A(\lambda) + B(t)\Big)u(t), \quad t \in \mathbb{R}, \tag{6.4}$$

where

$$A(\lambda)=\begin{bmatrix}0 & \Gamma\\ \lambda-A_0 & c\Gamma\end{bmatrix}\quad\text{and}\quad B(t)=\begin{bmatrix}0 & 0\\ -B_0(t) & -B_1(t)\end{bmatrix}. \tag{6.5}$$

Next, we will show that the operator $A(\lambda)$ defined on $\mathrm{dom}(A_0)\times\mathrm{dom}(|A_0\Gamma|^{1/2})$, has discrete spectrum, for λ in a large open subset of $\mathbb{C}$, and it is the generator of an analytic stable bi-semigroup on the space $\mathcal{X}=\mathrm{dom}(|A_0\Gamma^{-1}|^{1/2})\times\mathcal{X}_0$.

Lemma 6.3. *Assume Hypothesis 6.1. The following assertions are true:*

(i) The spectrum of $A(\lambda)$ consists of isolated semi–simple eigenvalues. Moreover,

$$\sigma(A(\lambda))=\{a_n^{\pm}(\lambda):n\in\mathbb{Z}_+\}\quad\textit{where}\quad a_n^{\pm}(\lambda)=\frac{c\gamma_n}{2}\pm\frac{1}{2}\Big(c^2\gamma_n^2+4\gamma_n(\lambda-\alpha_n)\Big)^{1/2}; \tag{6.6}$$

(ii) $\lim\limits_{n\to\infty}\frac{a_n^{\pm}(\lambda)}{\left(-\alpha_n\gamma_n\right)^{1/2}}=\pm1$ *for all* $\lambda\in\mathbb{C}$.

Proof. Assertion (i) follows from the fact that $A(\lambda)$ is similar to the operator of multiplication by the sequence of matrices (see Lemma A.3)

$$\begin{bmatrix}0 & \gamma_n\\ \lambda-\alpha_n & c\gamma_n\end{bmatrix},\quad n\in\mathbb{Z}_+,$$

whose eigenvalues are $a_n^{\pm}(\lambda)$, $n\in\mathbb{Z}_+$. Assertion (ii) follows from (A_3). □

In what follows we will construct an open set $\Omega(\mathcal{L})\subset\mathbb{C}$ such that $A(\lambda)$, for each $\lambda\in\Omega(\mathcal{L})$, generates a stable bi-semigroup on $\mathcal{X}$, and such that $a_n^{\pm}(\cdot)$ is holomorphic on $\Omega(\mathcal{L})$. We define

$$S_1(\mathcal{L})=\overline{\{\alpha_n-c^2\gamma_n/4+s:n\in\mathbb{Z}_+,s\in\mathbb{R}_-\}}, \tag{6.7}$$

$$S_2(\mathcal{L})=\overline{\{\alpha_n-ci\xi-\gamma_n^{-1}\xi^2:n\in\mathbb{Z}_+,\xi\in\mathbb{R}\}}, \tag{6.8}$$

and let

$$\Omega(\mathcal{L})=\mathbb{C}\setminus\big(S_1(\mathcal{L})\cup S_2(\mathcal{L})\big). \tag{6.9}$$

Lemma 6.4. *Assume Hypothesis 6.1. The following assertions hold true:*

(i) For any $\lambda\in\Omega(\mathcal{L})$ the operator $A(\lambda)$ defined in (6.5) *is the generator of an analytic, stable bi-semigroup. In particular the eigenvectors of $A(\lambda)$ form an orthonormal basis in $\mathcal{X}$;*

(ii) The functions $a_n^{\pm}(\cdot)$ are holomorphic on $\Omega(\mathcal{L})$ for any $n\in\mathbb{Z}_+$;

(iii) For each $\lambda_0\in\Omega(\mathcal{L})$ there exist $\varepsilon_0>0$ and $c_0>0$ such that

$$|\mathrm{Re}(a_n^{\pm}(\lambda))|\geq c_0\quad\textit{for all}\quad\lambda\in B(\lambda_0,\varepsilon_0),\ n\in\mathbb{Z}_+.$$

Proof. (i) We fix $\lambda\in\Omega(\mathcal{L})$ and define $A_0(\lambda)=\lambda-A_0$. Let us apply Lemma A.3 for $A_0(\lambda)$ and Γ. From Hypothesis 6.1 we infer that to prove that $A(\lambda)$ is the generator of a stable bi-semigroup, it is enough to show that the operators $A_0(\lambda)+\frac{c^2}{4}\Gamma$ and $A_0(\lambda)+ci\xi+\xi^2\Gamma^{-1}$ are invertible for all $\xi\in\mathbb{R}$.

Since $\lambda\in\Omega(\mathcal{L})$ we have that $\lambda\notin S_1(\mathcal{L})$ which implies that

$$\lambda-\alpha_n+\frac{c^2}{4}\gamma_n\neq0\quad\text{for all}\quad n\in\mathbb{Z}_+.$$

Moreover, from (A_3) we deduce

$$\lim_{n\to\infty}\Big|\lambda-\alpha_n+\frac{c^2}{4}\gamma_n\Big|=\infty,\ \Big(A_0(\lambda)+\frac{c^2}{4}\Gamma\Big)e_n=\Big(\lambda-\alpha_n+\frac{c^2}{4}\gamma_n\Big)e_n\ \text{for all}\ n\in\mathbb{Z}_+,$$

which implies that $A_0(\lambda)+\frac{c^2}{4}\Gamma$ is invertible. Similarly, for $\lambda\in\Omega(\mathcal{L})$ we have that $\lambda\notin S_2(\lambda)$ and so,

$$\lambda-\alpha_n+ci\xi+\gamma_n^{-1}\xi^2\neq 0 \quad \text{for all} \quad n\in\mathbb{Z}_+,\xi\in\mathbb{R}.$$

In addition, using again (A_3) we infer that

$$\lim_{n\to\infty}\Big|\lambda-\alpha_n+ci\xi+\gamma_n^{-1}\xi^2\Big|=\infty,\ \Big(A_0(\lambda)+ci\xi+\xi^2\Gamma^{-1}\Big)e_n=\Big(\lambda-\alpha_n+ci\xi+\gamma_n^{-1}\xi^2\Big)e_n$$

for all $n\in\mathbb{Z}_+,\xi\in\mathbb{R}$, which proves that $A_0(\lambda)+ci\xi+\xi^2\Gamma^{-1}$ is invertible for all $\xi\in\mathbb{R}$. From Lemma A.3 it follows that $A(\lambda)$ generates a stable bi-semigroup.

From Lemma 6.3(ii) it follows that $A(\lambda)$ is a bi-sectorial, multiplication operator, proving that the bi-semigroup generated by $A(\lambda)$ is analytic.

(ii) This assertion follows since $\Omega(\mathcal{L})\subseteq\mathbb{C}\setminus S_1(\mathcal{L})$, which implies that

$$\frac{c^2}{4}\gamma_n+\lambda-\alpha_n\in\mathbb{C}\setminus\mathbb{R}_- \quad \text{for all} \quad \lambda\in\Omega(\mathcal{L}).$$

(iii) From the definition of $a_n^{\pm}(\cdot)$ it follows that

$$\frac{a_n^{\pm}(\lambda)}{(-\alpha_n\gamma_n)^{1/2}}\to\pm1 \quad \text{as} \quad n\to\infty,$$

uniformly in $\lambda\in B(\lambda_0,\varepsilon_0)$, which takes care of the estimate for large $n\in\mathbb{Z}_+$. Since, in addition, $\mathrm{Re}(a_n^{\pm}(\lambda_0))\neq 0$, we infer that we can choose $\varepsilon_0>0$ small enough so that the estimate holds for small $n\in\mathbb{Z}_+$. □

To construct the Evans function using the determinant introduced in (3.29) we have to ensure that $A(\lambda)$ satisfies Hypothesis 2.2 for all $\lambda\in\Omega(\mathcal{L})$ and $B(\cdot)$ satisfies Hypothesis 3.2. We know that $\mathrm{Re}(a_n^{\pm}(\lambda))\neq 0$ for each $n\in\mathbb{Z}_+$, $\lambda\in\Omega(\mathcal{L})$ by Lemma 6.4. We re-denote by $a_j(\lambda)$ with $j\in\mathbb{Z}_+$ those $a_n^{\pm}(\lambda)$ for which $\mathrm{Re}(a_n^{\pm}(\lambda))>0$ and by $-a_j(\lambda)$ with $j\in\mathbb{Z}_-$ those $a_n^{\pm}(\lambda)$ for which $\mathrm{Re}(a_n^{\pm}(\lambda))<0$. Thus, as in (2.1),

$$\mathrm{Re}(a_j(\lambda))>0 \quad \text{for each} \quad j\in\mathbb{Z}\setminus\{0\},\ \lambda\in\Omega(\mathcal{L}). \tag{6.10}$$

We define the sequence $(\tilde{a}_n)_{n\in\mathbb{Z}\setminus\{0\}}$ by the formula

$$\tilde{a}_n=\begin{cases}(-\alpha_n\gamma_n)^{1/2}+\mathrm{Re}(\frac{c\gamma_n}{2}), & n\geq 1,\\ -(-\alpha_{-n}\gamma_{-n})^{1/2}+\mathrm{Re}(\frac{c\gamma_{-n}}{2}), & n\leq -1.\end{cases} \tag{6.11}$$

Since $\tilde{a}_n\to\pm\infty$ as $n\to\pm\infty$ we have that $\{\tilde{a}_n:n\in\mathbb{Z}\setminus\{0\}\}$ is a discrete set. Thus, we can rearrange its terms to obtain an increasing sequence $(\tilde{\varkappa}_j)_{j\in\mathbb{Z}\setminus\{0\}}$. Here, we also recall the definition of the sequence $(\varkappa_j(\lambda))_{j\in\mathbb{Z}\setminus\{0\}}$, associated to $A(\lambda)$, given in (2.12).

From Hypothesis 6.1 we infer that $\mathrm{Re}(a_n^{+}(\lambda))-\tilde{a}_n\to 0$ and $\mathrm{Re}(a_n^{-}(\lambda))-\tilde{a}_{-n}\to 0$ as $n\to\infty$, uniformly on bounded subsets of $\Omega(\mathcal{L})$, which implies that

$$\psi_j(\lambda):=\varkappa_j(\lambda)-\tilde{\varkappa}_j\to 0 \quad \text{as} \quad j\to\pm\infty \tag{6.12}$$

uniformly on bounded subsets of $\Omega(\mathcal{L})$.

Lemma 6.5. *Assume Hypothesis 6.1. Then, the sequence $(\tilde{\varkappa}_j)_{j\in\mathbb{Z}\setminus\{0\}}$ satisfies Hypothesis 2.2 if and only if $(\varkappa_j(\lambda))_{j\in\mathbb{Z}\setminus\{0\}}$ satisfies Hypothesis 2.2 for one/all $\lambda\in\Omega(\mathcal{L})$.*

Proof. Fix $\lambda \in \Omega(\mathcal{L})$ and assume that $(\tilde{\varkappa}_j)_{j\in\mathbb{Z}\setminus\{0\}}$ satisfies Hypothesis 2.2. Then,

$$\frac{\varkappa_j(\lambda)-\varkappa_k(\lambda)}{\tilde{\varkappa}_j-\tilde{\varkappa}_k} = 1 + \frac{\psi_j(\lambda)-\psi_k(\lambda)}{\tilde{\varkappa}_j-\tilde{\varkappa}_k} \quad \text{for all} \quad k \neq j.$$

Since $(\tilde{\varkappa}_j)_{j\in\mathbb{Z}\setminus\{0\}}$ satisfies Hypothesis 2.2 we have that there exists $M > 0$ such that

$$\frac{1}{|\tilde{\varkappa}_j-\tilde{\varkappa}_k|} \leq M \quad \text{for all} \quad k \neq j.$$

Using (6.12), we conclude that

$$\frac{\varkappa_j(\lambda)-\varkappa_k(\lambda)}{\tilde{\varkappa}_j-\tilde{\varkappa}_k} \to 1 \quad \text{as} \quad j,k\to\infty, k\neq j,$$

which proves the "only if" part of the lemma. The proof of the "if" part is analogous. □

This lemma shows that in order to ensure that $A(\lambda)$ satisfies Hypothesis 2.2 for all $\lambda \in \Omega$ we need to impose the following assumption.

Hypothesis 6.6.

$$\sum_{(k,j)\in\mathbb{Z}_{2,\pm}} (\tilde{\varkappa}_j-\tilde{\varkappa}_k)^{-2} < \infty.$$

We infer from (6.10) and Lemma 6.4 (iii) that $A(\lambda)$ satisfies (2.1) for each $\lambda \in \Omega(\mathcal{L})$. Since the operator Γ is invertible by (A_2), there exists a $c > 0$ such that $|\gamma_n| \geq c$ for each $n \in \mathbb{Z}_+$. Using (A_3), we conclude that $A(\lambda)$ satisfies (2.2) for each $\lambda \in \Omega(\mathcal{L})$ for $\gamma = \frac{2}{\nu}+1$. Thus, all the objects introduced in the previous sections for the bi-semigroup generator A can be defined for $A(\lambda)$ for all $\lambda \in \Omega(\mathcal{L})$, provided Hypothesis 6.6 holds. In what follows we are going to add λ–dependence to all of these objects.

Recall the definition of $\mathbb{Z}_{2,\pm}$ in (2.17). To ensure that $B(\cdot)$ in (6.5) satisfies Hypothesis 3.2 it is enough to assume the following.

Hypothesis 6.7. *The operator valued functions in* (6.2) *satisfy the properties*

$$\int_{\mathbb{R}_\pm} \|B_0(t)\|_2\,dt + \int_{\mathbb{R}_\pm} \|B_1(t)\|_2\,dt < 1.$$

Next, we recall a result (that dates back to D. Henry and was refined by B. Sandstede and A. Scheel in [34, 35]) that characterizes the discrete spectrum of the operator $\mathcal{L}$ from (6.2) in terms of the first order operator $\mathcal{T}(\lambda) : H^1(\mathbb{R}, \mathrm{dom}(A(\lambda))) \to L^2(\mathbb{R}, \mathcal{X})$ defined by

$$\mathcal{T}(\lambda) = \frac{d}{dt} - A(\lambda) - B(t), \tag{6.13}$$

where A and B are defined in (6.5). We recall definitions (2.19) (cf. (2.18) and (2.10)) of the integral operator $\mathcal{K} = \mathcal{K}(\lambda)$ associated with the bi-semigroup generator $A(\lambda)$.

Lemma 6.8. *Assume Hypotheses 6.1, 6.2, 6.6 and 6.7. The following assertions are true. For any* $\lambda \in \Omega(\mathcal{L})$*:*

(i) $\mathcal{T}(\lambda)$ *is Fredholm with index* 0*;*

(ii) $\mathcal{L} - \lambda$ *is Fredholm with index* 0*;*

(iii) $\lambda \in \sigma_{\mathrm{d}}(\mathcal{L})$ *if and only if* $\ker \mathcal{T}(\lambda) \neq \{0\}$*;*

(iv) $\lambda \in \sigma_{\mathrm{d}}(\mathcal{L})$ *if and only if* $\det_2(I - \mathcal{K}(\lambda)) = 0$.

Proof. (i) From Hypotheses 6.1 and 6.7 we infer that $B(t) \in \mathcal{B}_2(\mathcal{X})$ for all $t \in \mathbb{R}$ and $\|B(\cdot)\|_2 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$. Let $G(\lambda)$ be the operator defined using equation (2.36) but with $R(\cdot)$ replaced by $R(2\pi i\cdot, A(\lambda))$. Since $A(\lambda)$ is the generator of a stable bi-semigroup for all $\lambda \in \Omega(\mathcal{L})$, by Lemma 6.4, from Lemma 2.5 it follows that the operator $G(\lambda)$ is Fredholm with index 0. Moreover, from [25, Lemma 2.1] we have that $\mathcal{T}(\lambda) = G(\lambda)$, thus proving (i).

The proof of (ii) and (iii) follows closely the proof of [35, Theorem A.1.]. Fix $\lambda \in \Omega(\mathcal{L})$. First, we note that since $A(\lambda)$ is the generator of a stable bi-semigroup, the linear operator $\mathcal{T}_{\rm ref}(\lambda) : H^1(\mathbb{R}, \mathrm{dom}(A(\lambda))) \to L^2(\mathbb{R}, \mathcal{X})$ defined by $\mathcal{T}_{\rm ref}(\lambda) = \frac{d}{dt} - A(\lambda)$ is invertible. We define the linear operator $\mathcal{P} : L^2(\mathbb{R}, \mathcal{X}) \to L^2(\mathbb{R}, \mathcal{X}_0)$ by $\mathcal{P}(u, v)^{\rm T} = B_0(\cdot)u(\cdot) + B_1(\cdot)v(\cdot)$. Next, we observe that

$$F \in \mathrm{im}(\mathcal{T}(\lambda)) \quad \text{if and only if} \quad \Big(0, \mathcal{P}\big(\mathcal{T}_{\rm ref}(\lambda)\big)^{-1} F\Big)^{\rm T} \in \mathrm{im}(\mathcal{T}(\lambda));$$

$$f \in \mathrm{im}(\mathcal{L} - \lambda) \quad \text{if and only if} \quad (0, f)^{\rm T} \in \mathrm{im}(\mathcal{T}(\lambda)).$$

Using this equivalences one can easily show that $\mathrm{im}(\mathcal{L} - \lambda)$ is closed if and only if $\mathrm{im}(\mathcal{T}(\lambda))$ is closed. A direct computation shows that $\ker(\mathcal{L} - \lambda) \cong \ker(\mathcal{T}(\lambda))$ and $\ker(\mathcal{L} - \lambda)^* \cong \ker(\mathcal{T}(\lambda)^*)$, proving (ii) and (iii).

The proof of (iv) is a special case of the Birman–Schwinger principle. For completeness we give the details. From [25, Prop 7.3, 7.4] and (iii) we conclude that $\lambda \in \sigma_{\rm d}(\mathcal{L})$ if and only if $\ker(I - \mathcal{V}(\lambda) * M_B) \neq \{0\}$. From Lemma 2.5 we know that $\mathcal{V}(\lambda) * M_B \in \mathcal{B}_2(L^2(\mathbb{R}, \mathcal{X}))$ and thus $\lambda \in \sigma_{\rm d}(\mathcal{L})$ if and only if $\det_2(I - \mathcal{V}(\lambda) * M_B) = 0$. Finally, using that $\det_2(I - T_1T_2) = \det_2(I - T_2T_1)$ for any bounded linear operators T_1 and T_2 with $T_1T_2 \in \mathcal{B}_2$, we conclude that $\det_2(I - \mathcal{V}(\lambda) * M_B) = \det_2(I - \mathcal{K}(\lambda))$, proving the lemma. □

Remark 6.9. Assume Hypothesis 6.1. If $\lambda, \mu \in \Omega(\mathcal{L})$ and $z \in \rho(A(\lambda)) \cap \rho(A(\mu))$ then the following assertion is true:

$$R(z, A(\lambda)) - R(z, A(\mu)) = (\lambda - \mu) R(z, A(\lambda)) \mathcal{J} R(z, A(\mu)), \tag{6.14}$$

where $\mathcal{J} = \left[\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right]$.

Next, we will show that the functions $\mathcal{V}(\cdot)$ and $\mathcal{K}(\cdot)$ defined by (2.10) and (2.19) with A replaced by $A(\lambda)$ from (6.5) are holomorphic on $\Omega(\mathcal{L})$.

Lemma 6.10. *Assume Hypotheses 6.1, 6.2, 6.6 and 6.7. The following assertions are true:*

(i) $\mathcal{V} : \Omega(\mathcal{L}) \to C_{\rm b}(\mathbb{R}, \mathcal{B}(\mathcal{X}))$ *is holomorphic;*

(ii) $\mathcal{K} : \Omega(\mathcal{L}) \to \mathcal{B}_2(L^2(\mathbb{R}, \mathcal{X}))$ *is holomorphic.*

Proof. Fix $\lambda_0 \in \Omega(\mathcal{L})$ and let $\varepsilon_0 > 0$ be such that $B(\lambda_0, \varepsilon_0) \subseteq \Omega(\mathcal{L})$. Since $A(\lambda)$ generates a stable bi-semigroup we have

$$\sigma(A(\lambda)) \cap i\mathbb{R} = \emptyset \quad \text{for all} \quad \lambda \in B(\lambda_0, \varepsilon_0). \tag{6.15}$$

In addition, since the bi-semigroup generated by $A(\lambda)$ is analytic, there is a $c > 0$ such that

$$\|R(2\pi i\xi, A(\lambda))\| \leq \frac{c}{1 + 2\pi|\xi|} \quad \text{for all} \quad \xi \in \mathbb{R}, \lambda \in B(\lambda_0, \varepsilon_0). \tag{6.16}$$

Define $\mathcal{W} = \mathcal{F}^{-1}\Big(R(2\pi i\cdot, A(\lambda_0)) \mathcal{J} R(2\pi i\cdot, A(\lambda_0))\Big)$, where $\mathcal{F}$ is the Fourier transform. Since $\mathcal{F}\mathcal{W} \in L^1(\mathbb{R}, \mathcal{B}(\mathcal{X})) \cap L^2(\mathbb{R}, \mathcal{B}(\mathcal{X}))$ we have that $\mathcal{W} \in C_{\rm b}(\mathbb{R}, \mathcal{B}(\mathcal{X}))$

and

$$\|\mathcal{W}\|_\infty \leq \|\mathcal{F}\mathcal{W}\|_1 \leq c\int_{\mathbb{R}} \frac{d\xi}{(1+2\pi|\xi|)^2} < \infty.$$

Using Remark 6.9 and the fact that $\mathcal{F}\mathcal{V}(\lambda) = R(2\pi i\cdot, A(\lambda))$ for all $\lambda \in B(\lambda_0, \varepsilon_0)$, we compute:

$$\begin{aligned}
&\frac{1}{\lambda-\lambda_0}\big(\mathcal{V}(\lambda)-\mathcal{V}(\lambda_0)\big)-\mathcal{W} = \\
&\qquad = \mathcal{F}^{-1}\Big[\big(R(2\pi i\cdot, A(\lambda)) - R(2\pi i\cdot, A(\lambda_0))\big)\mathcal{J}R(2\pi i\cdot, A(\lambda_0))\Big] \\
&\qquad = (\lambda-\lambda_0)\mathcal{F}^{-1}\Big[R(2\pi i\cdot, A(\lambda))\mathcal{J}R(2\pi i\cdot, A(\lambda_0))\mathcal{J}R(2\pi i\cdot, A(\lambda_0))\Big], \qquad (6.17)
\end{aligned}$$

for all $\lambda \in B(\lambda_0,\varepsilon_0)\setminus\{\lambda_0\}$. From (6.16) it follows that

$$\mathcal{F}\Big[\frac{1}{\lambda-\lambda_0}\big(\mathcal{V}(\lambda)-\mathcal{V}(\lambda_0)\big)-\mathcal{W}\Big] \in L^1(\mathbb{R},\mathcal{B}(\mathcal{X}))\cap L^2(\mathbb{R},\mathcal{B}(\mathcal{X})),$$

and thus $\frac{1}{\lambda-\lambda_0}\big(\mathcal{V}(\lambda)-\mathcal{V}(\lambda_0)\big)-\mathcal{W} \in C_{\mathrm{b}}(\mathbb{R},\mathcal{B}(\mathcal{X}))$ and

$$\begin{aligned}
\Big\|\frac{1}{\lambda-\lambda_0}\big(\mathcal{V}(\lambda)-\mathcal{V}(\lambda_0)\big)-\mathcal{W}\Big\|_\infty &\leq \Big\|\mathcal{F}\Big[\frac{1}{\lambda-\lambda_0}\big(\mathcal{V}(\lambda)-\mathcal{V}(\lambda_0)\big)-\mathcal{W}\Big]\Big\|_1 \\
&\leq c|\lambda-\lambda_0|\int_{\mathbb{R}}\frac{d\xi}{(1+2\pi|\xi|)^3} = c|\lambda-\lambda_0| \quad \text{for all} \quad \lambda\in B(\lambda_0,\varepsilon_0)\setminus\{\lambda_0\}.
\end{aligned} \tag{6.18}$$

It follows that $\mathcal{V}$ is differentiable at λ_0, proving assertion (i). Since $\mathcal{K}(\lambda) = M_{B_r}\big(\mathcal{V}(\lambda) * M_{B_\ell}\big)$ for all $\lambda \in \Omega(\mathcal{L})$, and since $B_r(t), B_\ell(t) \in \mathcal{B}_4(\mathcal{X})$ and $\|B_\ell(t)\|_4 = \|B_r(t)\|_4 = \|B(t)\|_2^{1/2}$ for all $t\in\mathbb{R}$, assertion (ii) follows shortly from (i), Hypothesis 6.2 and Hypothesis 6.7, arguing similarly to Lemma 2.3. □

In the next lemma we prove that the projections $\mathcal{Q}_\pm^{(n)}(\cdot)$ defined by (2.25) with A replaced by $A(\lambda)$, are holomorphic for n large enough.

Lemma 6.11. *Assume Hypotheses 6.1, 6.6 and let $\lambda_0 \in \Omega(\mathcal{L})$. Then there exists $\varepsilon_0 > 0$ and $N = N(\lambda_0) \in \mathbb{Z}_+$ such that $\mathcal{Q}_\pm^{(n)}(\cdot)$ is holomorphic on $B(\lambda_0,\varepsilon_0)$ for all $n \geq N$.*

Proof. By Lemma 6.4 we can choose $\varepsilon_0 > 0$ small enough such that $B(\lambda_0,\varepsilon_0) \subseteq \Omega(\mathcal{L})$ and

$$|\mathrm{Re}(a_n^\pm(\lambda))| \geq c_0 > 0 \quad \text{for all} \quad \lambda \in B(\lambda_0,\varepsilon_0),\ n\in\mathbb{Z}_+. \tag{6.19}$$

In addition, since $A(\lambda)$ is the generator of an analytic bi-semigroup, there exists $\theta \in (0,\pi/2)$ such that

$$\sigma(A(\lambda)) \subseteq \Big(\Sigma_\theta^+ \cup \Sigma_\theta^-\Big) \cap \{z\in\mathbb{C} : |\mathrm{Re}\, z| \geq c_0\} \tag{6.20}$$

for all $\lambda \in B(\lambda_0,\varepsilon_0)$, where

$$\Sigma_\theta^+ = \{z\in\mathbb{C} : \pi-\theta < \arg(z) < \pi+\theta\}, \quad \Sigma_\theta^- = \{z\in\mathbb{C} : -\theta < \arg(z) < \theta\}. \tag{6.21}$$

From (6.12) it follows that

$$|\varkappa_j(\lambda)-\varkappa_j(\lambda_0)| \leq |\varkappa_j(\lambda)-\tilde{\varkappa}_j| + |\varkappa_j(\lambda_0)-\tilde{\varkappa}_j| \to 0 \quad \text{as} \quad j\to\pm\infty,$$

uniformly on $B(\lambda_0,\varepsilon_0)$. Next, we choose $N_1 = N_1(\lambda_0)\in\mathbb{Z}_+$ such that

$$|\varkappa_j(\lambda)-\varkappa_j(\lambda_0)| < 1/2 \quad \text{for all} \quad \lambda\in B(\lambda_0,\varepsilon_0),\ j\in\mathbb{Z} \quad \text{with} \quad |j|\geq N_1. \tag{6.22}$$

By the definition of $\psi_j(\cdot)$ in (6.12) we have

$$\varkappa_{j+1}(\lambda) - \varkappa_j(\lambda) = \tilde{\varkappa}_{j+1} - \tilde{\varkappa}_j + \psi_{j+1}(\lambda) - \psi_j(\lambda) \tag{6.23}$$

for all $\lambda \in B(\lambda_0, \varepsilon_0)$, $j \in \mathbb{Z}$. From Hypothesis 6.6 we obtain that

$$\tilde{\varkappa}_{j+1} - \tilde{\varkappa}_j \to \infty \quad \text{as} \quad j \to \pm\infty. \tag{6.24}$$

Since

$$\psi_{j+1}(\lambda) \to 0, \quad \psi_j(\lambda) \to 0 \quad \text{as} \quad j \to \pm\infty. \tag{6.25}$$

uniformly on $B(\lambda_0, \varepsilon_0)$, by (6.12), from (6.23), (6.24) and (6.25) we conclude that there exists $N_2 = N_2(\lambda_0) \in \mathbb{Z}_+$ such that

$$\varkappa_{j+1}(\lambda) - \varkappa_j(\lambda) > 2 \quad \text{for all} \quad \lambda \in B(\lambda_0, \varepsilon_0),\ j \in \mathbb{Z} \quad \text{with} \quad |j| \geq N_2. \tag{6.26}$$

Define $N = N(\lambda_0) = \max\{N_1(\lambda_0), N_2(\lambda_0)\} \in \mathbb{Z}_+$. Fix $n \in \mathbb{Z}_+$ such that $n \geq N$. From (6.22) and (6.26) we estimate

$$\begin{aligned}\varkappa_{-n-1}(\lambda) &\leq \varkappa_{-n}(\lambda) - 2 < (\varkappa_{-n}(\lambda_0) + 1/2) - 2 = \varkappa_{-n}(\lambda_0) - 3/2 \\ &< \varkappa_{-n}(\lambda_0) - 1 < \varkappa_{-n}(\lambda_0) - 1/2 < \varkappa_{-n}(\lambda).\end{aligned}$$

That is

$$\begin{aligned}&\varkappa_j(\lambda) < \varkappa_{-n}(\lambda_0) - 1 \quad \text{for all} \quad \lambda \in B(\lambda_0, \varepsilon_0),\ j \leq -n-1, \\ &\varkappa_j(\lambda) > \varkappa_{-n}(\lambda_0) - 1 \quad \text{for all} \quad \lambda \in B(\lambda_0, \varepsilon_0),\ -n \leq j \leq -1.\end{aligned} \tag{6.27}$$

Choose $\delta \in (\theta, \pi/2)$ and let C_n^+ be the anti-clockwise oriented path, along the sides of the triangle $\arg(z) = \pi + \delta$, $\arg(z) = \pi - \delta$, $\mathrm{Re}\,(z) = \varkappa_{-n}(\lambda_0) - 1$. From (6.20) and (6.27) we infer that

$$\mathcal{Q}_+^{(n)}(\lambda) = \frac{1}{2\pi i} \oint_{C_n^+} R(z, A(\lambda))\, dz \quad \text{for all} \quad \lambda \in B(\lambda_0, \varepsilon_0). \tag{6.28}$$

Using this integral representation and (6.14) one can easily show that $\mathcal{Q}_+^{(n)}(\cdot)$ is holomorphic on $B(\lambda_0, \varepsilon_0)$. Similarly to (6.27), from (6.22) and (6.26) we have that

$$\begin{aligned}&\varkappa_j(\lambda) > \varkappa_n(\lambda_0) + 1 \quad \text{for all} \quad \lambda \in B(\lambda_0, \varepsilon_0),\ j \geq n+1 \\ &\varkappa_j(\lambda) < \varkappa_n(\lambda_0) + 1 \quad \text{for all} \quad \lambda \in B(\lambda_0, \varepsilon_0),\ 1 \leq j \leq n.\end{aligned} \tag{6.29}$$

Then define C_n^- as the anti-clockwise oriented path, along the sides of the triangle $\arg(z) = -\delta$, $\arg(z) = \delta$, $\mathrm{Re}\,(z) = \varkappa_n(\lambda_0) + 1$. From (6.20) and (6.29) it follows that

$$\mathcal{Q}_-^{(n)}(\lambda) = \frac{1}{2\pi i} \oint_{C_n^-} R(z, A(\lambda))\, dz \quad \text{for all} \quad \lambda \in B(\lambda_0, \varepsilon_0). \tag{6.30}$$

Using again (6.14) we obtain that $\mathcal{Q}_-^{(n)}(\cdot)$ is holomorphic on $B(\lambda_0, \varepsilon_0)$. □

In the next lemma we show the existence and analyticity of certain solutions of the truncated equation (6.4), that is of (2.38) with A replaced by $A(\lambda)$. Specifically, in the sequel we consider the equations

$$\begin{aligned}Y^\lambda_{+,n,V}(t) = T^\lambda_+(t)\mathcal{Q}_+^{(n)}(\lambda) - \int_t^\infty \varphi_n^2(s)\Big[T^\lambda_+(t-s)\mathcal{Q}_+^{(n)}(\lambda) \\ + T^\lambda_-(s-t)\mathcal{Q}_-^{(n)}(\lambda)\Big] B(s)\mathcal{Q}^{(n)}(\lambda) Y^\lambda_{+,n,V}(s)\, ds, \quad t \geq 0,\end{aligned} \tag{6.31}$$

$$Y^\lambda_{-,n,V}(t) = T^\lambda_-(-t)\mathcal{Q}_-^{(n)}(\lambda) + \int_{-\infty}^t \varphi_n^2(s)\Big[T^\lambda_+(t-s)\mathcal{Q}_+^{(n)}(\lambda)$$

$$+ T_-^\lambda(s-t)\mathcal{Q}_-^{(n)}(\lambda)\Big] B(s)\mathcal{Q}^{(n)}(\lambda)Y_{-,n,V}^\lambda(s)\,ds, \quad t \le 0. \tag{6.32}$$

This system of Volterra integral equation is constructed similarly to the truncated Fredholm integral equations (4.1), (4.2). We use $\{T_\pm^\lambda(t)\}_{t\ge 0}$ to denote the semigroups (2.3) with a_n replaced with $a_n(\lambda)$, and we use subscript V to emphasize that solutions $Y_{\pm,n,V}^\lambda(\cdot)$ solve the Volterra integral equations.

Lemma 6.12. *Assume Hypotheses 6.6 and 6.7. The following assertions are true:*

(i) Equations (6.31) *and* (6.32) *have unique solutions in* $C_\mathrm{b}(\mathbb{R}_\pm, \mathcal{B}(\operatorname{im}\mathcal{Q}^{(n)}(\lambda)))$ *for all* $n \in \mathbb{Z}_+$ *and* $\lambda \in \Omega(\mathcal{L})$*;*

(ii) For any $\lambda_0 \in \Omega(\mathcal{L})$ *and the numbers* ε_0 *and* N *introduced in Lemma 6.11, the functions* $\mathcal{Y}_{\pm,n,V} : B(\lambda_0,\varepsilon_0) \to \mathcal{B}(\mathcal{X})$ *defined by* $\mathcal{Y}_{\pm,n,V}(\lambda) = Y_{+,n,V}^\lambda(0)$*, are holomorphic for all* $n \ge N$*.*

Proof. (i) The existence and uniqueness of solutions of the Volterra equations (6.31) and (6.32) follows directly from Hypothesis 6.7 and a contraction mapping argument. To prove (ii), we first fix $\lambda_0 \in \Omega(\mathcal{L})$, and consider $\varepsilon_0 > 0$ and $N = N(\lambda_0)$ given by Lemma 6.11 such that $\mathcal{Q}_\pm^{(n)}(\cdot)$ is holomorphic on $B(\lambda_0,\varepsilon_0)$ for all $n \ge N$. Moreover, from Lemma 6.10(i) we have that $\mathcal{V}$ is holomorphic on $\Omega(\mathcal{L})$ which implies that $\mathcal{V}(\cdot)\mathcal{Q}_\pm^{(n)}(\cdot)$ are holomorphic on $B(\lambda_0,\varepsilon_0)$. It follows that the functions $\tilde{T}_{\pm,n,\alpha} : B(\lambda_0,\varepsilon_0) \to C_\mathrm{b}([-\alpha,\alpha],\mathcal{B}(\mathcal{X}))$ defined by

$$\Big[\tilde{T}_{\pm,n,\alpha}(\lambda)\Big](t) = T_\pm^\lambda(t)\mathcal{Q}_\pm^{(n)}(\lambda)$$

are holomorphic for all $n \ge N$ and all $\alpha > 0$. Since the support of the functions $\varphi_n(\cdot)$ is compact, the solutions $Y_{\pm,n,V}^\lambda$ generate holomorphic functions $\lambda \mapsto Y_{\pm,n,V}^\lambda(\cdot) \in C_\mathrm{b}(\mathbb{R},\mathcal{B}(\mathcal{X}))$, and the lemma follows shortly. □

The next theorem is the main result of this section: it shows that the Evans determinant defined in (3.29) defined via the generalized Jost solutions is a holomorphic function, and that it detects the discrete eigenvalues of the operator $\mathcal{L}$.

Theorem 6.13. *Assume Hypotheses 6.1, 6.2, 6.6 and 6.7. Then, the following assertions are true:*

(i) If $\lambda \in \Omega(\mathcal{L})$ *then* $\lambda \in \sigma_\mathrm{d}(\mathcal{L})$ *if and only if* $E(\lambda) = 0$*;*

(ii) E *is holomorphic on* $\Omega(\mathcal{L})$*.*

Proof. Assertion (i) follows from Theorem 5.8 and Lemma 6.8(iv).

The proof of (ii) is divided into several steps. First, we observe from Theorem 5.8 that

$$E(\lambda) = e^{-\Xi(\lambda)} \det{}_{2,L^2(\mathbb{R},\mathcal{X})}(I - \mathcal{K}(\lambda)) \tag{6.33}$$

where

$$\Xi(\lambda) = \operatorname{tr}\left[\operatorname{diag}\left(\mathcal{Y}_+^\lambda + \mathcal{Y}_-^\lambda - I + \int_{\mathbb{R}_+} \mathcal{Q}_+(\lambda)B(t)\,dt - \int_{\mathbb{R}_-} \mathcal{Q}_-(\lambda)B(t)\,dt\right)\right]. \tag{6.34}$$

Here $\mathcal{Y}_\pm^\lambda$ are defined by (3.27), with A replaced with $A(\lambda)$ throughout Section 3. In particular, $\mathcal{Y}_\pm^\lambda$ are determined via the solutions $Y_\pm^{(j),\lambda}(\cdot)$ of the Fredholm type integral equations (3.1), (3.2) with A replaced by $A(\lambda)$. We note that since the trace does not depend on the choice of a Hilbert space basis, $\operatorname{tr}(\operatorname{diag}(T))$ does not depends on the choice of the Hilbert basis used to define $\operatorname{diag}(T)$.

In view of Lemma 6.10(ii), to finish the proof the theorem it is enough to show that the function $\Xi(\cdot)$ is holomorphic on $\Omega(\mathcal{L})$. To prove the analyticity of $\Xi(\cdot)$ we will approximate it by a sequence of holomorphic functions. We define $\mathbf{Z}_\pm(\lambda) = \{Z_\pm^{(j)}(\lambda)\}_{j\in\mathbb{Z}_\mp}$ using (3.10)–(3.12) with A replaced by $A(\lambda)$ and $\mathbf{Z}_{\pm,n}(\lambda) = \{Z_{\pm,n}^{(j)}(\lambda)\}_{j\in\mathbb{Z}_\mp}$ using (3.10)–(3.12) with A replaced by $A(\lambda)$. In addition, we introduce the operator valued function B_n^λ by the formula $B_n^\lambda(\cdot) = \varphi_n^2(\cdot)\mathcal{Q}^{(n)}(\lambda)B(\cdot)\mathcal{Q}^{(n)}(\lambda)$, $\lambda \in \Omega(\mathcal{L})$, $n \in \mathbb{Z}_+$. Finally, we introduce the functions $\Xi_n : \Omega(\mathcal{L}) \to \mathbb{C}$ by

$$\Xi_n(\lambda) = \operatorname{tr}\big(D^+(B_n^\lambda, B_n^\lambda, \mathbf{Z}_{+,n}(\lambda) + D^-(B_n^\lambda, B_n^\lambda, \mathbf{Z}_{-,n}(\lambda))\big), \tag{6.35}$$

where $D^\pm$ is defined in (5.18).

Next, we will prove that the sequence of functions $\Xi_n(\cdot)$ is uniformly bounded. From Hypothesis 6.7, (4.4) and Lemma 4.1(v), we have that

$$\|Z_{\pm,n}^{(j)}(\lambda)\|_{C_\mathrm{b}(\mathbb{R}_\pm,\mathcal{B}(\mathcal{X}))} \le c^\#\Big(\int_\mathbb{R} \|B_n^\lambda(t)\|_2\,dt\Big) \tag{6.36}$$

for all $j \in \mathbb{Z}_\mp$, $\lambda \in \Omega(\mathcal{L})$ and $n \in \mathbb{Z}_+$, where $c^\# : [0,1) \to \mathbb{R}_+$ is an increasing function. Since

$$\|B_n^\lambda(t)\|_2 \le \|B(t)\|_2 \tag{6.37}$$

for all $t \in \mathbb{R}$, $\lambda \in \Omega(\mathcal{L})$ and $n \in \mathbb{Z}_+$, we conclude by (6.36) that

$$\|Z_{\pm,n}^{(j)}(\lambda)\|_{C_\mathrm{b}(\mathbb{R}_\pm,\mathcal{B}(\mathcal{X}))} \le c \tag{6.38}$$

for some constant $c > 0$, uniformly for all $j \in \mathbb{Z}_\mp$, $\lambda \in \Omega(\mathcal{L})$ and $n \in \mathbb{Z}_+$. From Lemma 5.4, (6.38) and (6.37) it follows that

$$\begin{aligned}|\Xi_n(\lambda)| &\le \|D^+(B_n^\lambda, B_n^\lambda, \mathbf{Z}_{+,n}(\lambda))\|_1 + \|D^-(B_n^\lambda, B_n^\lambda, \mathbf{Z}_{-,n}(\lambda))\|_1\\ &\le c\sup_{j\in\mathbb{Z}_-}\|Z_{+,n}^{(j)}(\lambda))\|_{C_\mathrm{b}(\mathbb{R}_+,\mathcal{B}(\mathcal{X}))} + c\sup_{j\in\mathbb{Z}_+}\|Z_{-,n}^{(j)}(\lambda))\|_{C_\mathrm{b}(\mathbb{R}_-,\mathcal{B}(\mathcal{X}))} \le c\end{aligned} \tag{6.39}$$

for all $\lambda \in \Omega(\mathcal{L})$, $n \in \mathbb{Z}_+$.

Next, Theorem 5.5 and Lemma 5.7(i) imply

$$\Xi_n(\lambda) \to \Xi(\lambda) \quad \text{as} \quad n \to \infty \quad \text{for each} \quad \lambda \in \Omega(\mathcal{L}). \tag{6.40}$$

By Vitali's theorem it remains to prove that the functions Ξ_n are holomorphic for n large enough. We fix $\lambda_0 \in \Omega(\mathcal{L})$ and consider $\varepsilon_0 > 0$ and $N = N(\lambda_0)$ given by Lemma 6.11 such that $\mathcal{Q}_\pm^{(n)}(\cdot)$ is holomorphic on $B(\lambda_0, \varepsilon_0)$ for all $n \ge N$. From (5.6)–(5.7), (5.43) we infer

$$\begin{aligned}\Xi_n(\lambda) = \operatorname{tr}\Bigg[&\mathcal{Y}_{+,n}^\lambda + \mathcal{Y}_{-,n}^\lambda - Q^{(n)}(\lambda) + \int_{\mathbb{R}_+} \mathcal{Q}_+^{(n)}(\lambda)B(t)\,dt\\ &- \int_{\mathbb{R}_-} \mathcal{Q}_-^{(n)}(\lambda)B(t)\,dt\Bigg],\end{aligned} \tag{6.41}$$

for all $\lambda \in B(\lambda_0, \varepsilon_0)$ and all $n \ge N$. Here $\mathcal{Y}_{n,\pm}^\lambda$ are defined via (4.7) by replacing A with $A(\lambda)$. On the other hand, from the proof of Lemma 7.6 in [14] we know that one can replace the Fredholm solutions in equation (6.43) by the corresponding Volterra solutions, without changing the diagonal blocks of $\mathcal{Y}^{\lambda}_{\pm,n}$ and thus the following formula holds true:

$$\Xi_n(\lambda) = \operatorname{tr}\Bigg[\mathcal{Y}_{+,n,V}(\lambda) + \mathcal{Y}_{-,n,V}(\lambda) - Q^{(n)}(\lambda) + \int_{\mathbb{R}_+} \mathcal{Q}^{(n)}_+(\lambda)B(t)\,dt - \int_{\mathbb{R}_-} \mathcal{Q}^{(n)}_-(\lambda)B(t)\,dt\Bigg]. \tag{6.42}$$

for all $\lambda \in B(\lambda_0, \varepsilon_0)$, and all $n \geq N$. Lemma 6.11 and Lemma 6.12(ii) imply that

$$\Xi_n : \Omega(\mathcal{L}) \to \mathbb{C} \quad \text{is holomorphic on} \quad B(\lambda_0, \varepsilon_0) \quad \text{for all} \quad n \geq N. \tag{6.43}$$

From (6.40), (6.39) and (6.43) it follows that $\Xi(\cdot)$ is holomorphic on $B(\lambda_0, \varepsilon_0)$, proving the theorem. □

7. Examples

In this section we present some examples that lead to eigenvalue problems for which we can construct an analytic infinite–dimensional Evans function, as described in Section 6.

Example 7.1. We consider the equation, for $u = u(t, \xi, \eta)$,

$$\partial_t u - \partial^2_\eta \left(f(u) + \partial^4_\eta u - \gamma\partial^2_\eta\right) - \partial^2_\xi u = 0,\ t \geq 0,\ \xi \in \mathbb{R},\ \eta \in (-\pi, \pi). \tag{7.1}$$

Here $\gamma > 0$ and $f : \mathbb{R} \to \mathbb{R}$ is a smooth function with $f(0) = f'(0) = 0$. Using methods from [5, Section 3] we have that, if $\alpha \in (0, \frac{\gamma}{2})$, the forth–order equation

$$q^{(4)} - \gamma q'' + \alpha^2 q + f(q) = 0 \tag{7.2}$$

has an exponentially localized homoclinic solution, that is, there exist constant $C, \delta > 0$ and q^∞ such that

$$|q(s) - q^\infty| \leq Ce^{-\delta|s|} \quad \text{for all} \quad s \in \mathbb{R}. \tag{7.3}$$

One can easily check that $u_0(t, \xi, \eta) = q(\eta + \alpha\xi)$ is a solution of (7.1). Linearizing along u_0, one has $\partial_t u = \mathcal{L}u$, where

$$\mathcal{L} = \partial^2_\xi + \partial^6_\eta - \gamma\partial^4_\eta + \partial^2_\eta \left(f'(\eta + \alpha\xi)u\right),\ \xi \in \mathbb{R},\ \eta \in (-\pi, \pi). \tag{7.4}$$

Choosing $\mathcal{X}_0 = L^2(-\pi, \pi)$, we will show that the operator $\mathcal{L}$ is of the form (6.2). Indeed, we can write $\mathcal{L}$ as follows,

$$\mathcal{L} = \partial^2_\xi + (A_0 + B_0(\xi)), \tag{7.5}$$

where $A_0 : H^6(-\pi, \pi) \to L^2(-\pi, \pi)$ and $B_0(\xi) : H^3(-\pi, \pi) \to L^2(-\pi, \pi)$, $\xi \in \mathbb{R}$, are given by

$$A_0 = \partial^6_\eta - \gamma\partial^4_\eta,\ (B_0(\xi)u)(\eta) = \partial^2_\eta \left(f'(q(\eta + \alpha\xi))\right). \tag{7.6}$$

Taking $\Gamma = I$, we can readily check that conditions (A_1)–(A_3) are satisfied for $\alpha_n = -n^6 - \gamma n^4$, $\gamma_n = 1$. From Remark A.2 we infer that $\partial^2_\eta \in \mathcal{B}_2(H^3(-\pi, \pi), L^2(-\pi, \pi))$, which implies that the operator valued function B_0 defined above satisfies condition (B_0). Using (6.11) we verify that in this case Hypothesis 6.6 is satisfied. Assuming a smallness condition on f', one can check that Hypothesis 6.7 is satisfied, which allows us to construct an Evans by applying Theorem 6.13.

Example 7.2. The fifth–order Kadomtsev-Petviashvili equation, for $u = u(t, \xi, \eta)$, is

$$\partial_\eta \left(\partial_t u + 2u\partial_\eta u + \partial^3_\eta u - \partial^5_\eta\right) - \partial^2_\zeta u = 0,\ t \geq 0,\ \zeta,\ \eta \in \mathbb{R}. \tag{7.7}$$

Among the solutions of equation (7.7) are the so called line-solitons, which are traveling waves of the form $u_0(t, \eta, \zeta) = \phi(\eta + \alpha\zeta - ct)$. Plugging this ansatz in (7.7) and integrating the equation for ϕ twice, we obtain the fourth–order equation,

$$\phi^{(4)} - \phi'' + (c + \alpha^2)\phi = \phi^2. \tag{7.8}$$

To find solutions of this equation, we can use the result in [5]. However, if $c+\alpha^2 = \frac{36}{169}$ then (7.8) has the exact solution

$$\phi(s) = \frac{105}{338}\mathrm{sech}^4\left(\frac{s}{2\sqrt{13}}\right).$$

Writing equation (7.7) in the moving frame coordinates $\xi = \zeta - \frac{c}{\alpha}t \in \mathbb{R}$, and then linearizing it along u_0, we obtain the equation

$$\partial_\eta\Big(\partial_t u + 2\phi(\eta + \alpha\xi)\partial_\eta u + 2\phi'(\eta + \alpha\xi)u + \partial^3_\eta u - \partial^5_\eta - \omega\partial_\xi u\Big) - \partial^2_\xi u = 0, \tag{7.9}$$

where $t \geq 0$, $\xi, \eta \in \mathbb{R}$ and $\omega = \frac{c}{\alpha}$. To compute the spectrum of the linearization (7.9), we look first at the truncated problem on the infinite cylinder $(\xi, \eta) \in \mathbb{R} \times (-N, N)$, for N large, with Dirichelet boundary conditions. Integrating this equation with respect to η, we obtain the equation

$$\partial_t u = \mathcal{L}u, \ \text{ where } \mathcal{L}u := \Gamma^{-1}\partial^2_\xi u + \omega\partial_\xi u + \partial^5_\eta u - \partial^3_\eta u - 2\phi(\eta+\alpha\xi)\partial_\eta u - 2\phi'(\eta+\alpha\xi)u,$$

Here $\Gamma : H^1_0(-N, N) \to L^2(-N, N)$ is defined by $\Gamma = \frac{1}{3}\partial_\eta$. We observe that the operator $\mathcal{L}$ is of the form (6.2), where $\mathcal{X}_0 = L^2(-N, N)$, Γ is defined above, and

$$c = -\omega,\ A_0 : H^5(-N, N) \cap H^1_0(-N, N) \to L^2(-N, N),\ A_0 = \partial^5_\eta - \partial^3_\eta. \tag{7.10}$$

Moreover, we have that $\mathrm{dom}(|A_0\Gamma^{-1}|^{1/2}) = H^2(-N, N) \cap H^1_0(-N, N)$, $B_1 \equiv 0$ and

$$(B_0(\xi)u)(\eta) = -2\phi(\eta + \alpha\xi)\partial_\eta u - 2\phi'(\eta + \alpha\xi)u. \tag{7.11}$$

We note that the operators A_0 and Γ satisfy Hypothesis 6.1 with $\gamma_n = \frac{in\pi}{3N}$ and $\alpha_n = \frac{in^5\pi^5}{N^5} + \frac{in^3\pi^3}{N^3}$. Since $\partial_\eta \in \mathcal{B}_2(H^2(-N, N), L^2(-N, N))$ and $(B_0(\xi)u)(\eta) = -2\partial_\eta(\phi(\eta + \alpha\xi)u)$, we infer that B_0 satisfies condition (B_0). Also, using (6.11) we compute that $\tilde{\varkappa}_j = \mathrm{sign}(j)\frac{j^2\pi^2\sqrt{j^2\pi^2+N^2}}{N^3\sqrt{3}}$. at this point, one can readily check that Hypothesis 6.6 and Hypothesis 6.7 are satisfied. Thus, we can construct an analytic Evans function via Theorem 6.13.

Appendix A. Miscellaneous

Lemma A.1. *Let $\mathcal{X}$ be a separable Hilbert space, $F : \mathbb{R} \to \mathcal{B}(\mathcal{X})$ be a strongly measurable function such that $F(t) \in \mathcal{B}_2(\mathcal{X})$ for all $t \in \mathbb{R}$ and $\int_\mathbb{R} \|F(t)\|_2\, dt < \infty$. The operator $T : \mathcal{X} \to \mathcal{X}$ defined by the formula $Tx = \int_\mathbb{R} F(t)x\, dt$, satisfies $T \in \mathcal{B}_2(\mathcal{X})$ and $\|T\|_2 \leq \int_\mathbb{R} \|F(t)\|_2\, dt$.*

Proof. Let $\{\mathrm{e}_n : n \in \mathbb{Z}_+\}$ be an orthonormal Hilbert basis in $\mathcal{X}$ and let $\mathcal{Q}^{(n)}$ be the orthogonal projection of $\mathcal{X}$ onto $\mathcal{X}_n = \mathrm{Span}\{\mathrm{e}_k : k = 1, \ldots, n\}$. If $n > m$ then

$\mathcal{Q}^{(n)}F(t)\mathcal{Q}^{(n)} - \mathcal{Q}^{(m)}F(t)\mathcal{Q}^{(m)}$ and $\mathcal{Q}^{(n)}T\mathcal{Q}^{(n)} - \mathcal{Q}^{(m)}T\mathcal{Q}^{(m)}$ are operators on $\mathcal{X}_n$ and

$$\mathcal{Q}^{(n)}T\mathcal{Q}^{(n)} - \mathcal{Q}^{(m)}T\mathcal{Q}^{(m)} = \int_{\mathbb{R}} \big(\mathcal{Q}^{(n)}F(t)\mathcal{Q}^{(n)} - \mathcal{Q}^{(m)}F(t)\mathcal{Q}^{(m)}\big)\, dt. \tag{A.1}$$

Since $\mathcal{X}_n$ is finite dimensional, the integral can be understood in the $\|\cdot\|_2$ sense and

$$\|\mathcal{Q}^{(n)}T\mathcal{Q}^{(n)} - \mathcal{Q}^{(m)}T\mathcal{Q}^{(m)}\|_2 \le \int_{\mathbb{R}} \|\mathcal{Q}^{(n)}F(t)\mathcal{Q}^{(n)} - \mathcal{Q}^{(m)}F(t)\mathcal{Q}^{(m)}\|_2\, dt.$$

Further, we recall from [37, Lem. 6.1.3] that if $K \in \mathcal{B}_2(\mathcal{X})$ and $\mathcal{Q}^{(n)} \to I$ strongly then $K\mathcal{Q}^{(n)} \to K$ and $\mathcal{Q}^{(n)}K \to K$ in $\mathcal{B}_2(\mathcal{X})$ as $n \to \infty$. Thus,

$$\begin{aligned}\|\mathcal{Q}^{(n)}F(t)\mathcal{Q}^{(n)} - \mathcal{Q}^{(m)}F(t)\mathcal{Q}^{(m)}\|_2 &\le \|F(t)\mathcal{Q}^{(n)} - F(t)\mathcal{Q}^{(m)}\|_2 \\ &\quad + \|\mathcal{Q}^{(n)}F(t) - \mathcal{Q}^{(m)}F(t)\|_2 \to 0\end{aligned}$$

as $n, m \to \infty$ for all $t \in \mathbb{R}$. Since $\|\mathcal{Q}^{(n)}F(t)\mathcal{Q}^{(n)} - \mathcal{Q}^{(m)}F(t)\mathcal{Q}^{(m)}\|_2 \le 2\|F(t)\|_2$, by (A.1) and Lebesgue's dominated convergence theorem $(\mathcal{Q}^{(n)}T\mathcal{Q}^{(n)})_{n\in\mathbb{Z}_+}$ is a Cauchy sequence in $\mathcal{B}_2(\mathcal{X})$. Since $\mathcal{Q}^{(n)}T\mathcal{Q}^{(n)} \to T$ strongly as $n \to \infty$, we conclude that $T \in \mathcal{B}_2(\mathcal{X})$ and $\mathcal{Q}^{(n)}T\mathcal{Q}^{(n)} \to T$ in $\mathcal{B}_2(\mathcal{X})$. Using the same argument as above one can see that

$$\|\mathcal{Q}^{(n)}T\mathcal{Q}^{(n)}\|_2 \le \int_{\mathbb{R}} \|\mathcal{Q}^{(n)}F(t)\mathcal{Q}^{(n)}\|_2\, dt \le \int_{\mathbb{R}} \|F(t)\|_2\, dt.$$

Passing to the limit as $n \to \infty$, we obtain that $\|T\|_2 \le \int_{\mathbb{R}} \|F(t)\|_2\, dt$. □

Remark A.2. Let $\mathcal{X}_0$ be a Hilbert space and $A_0 : \mathrm{dom}(A_0) \subseteq \mathcal{X}_0 \to \mathcal{X}_0$ be an unbounded positive definite operator. We equip $\mathrm{dom}(A_0)$ with the graph norm and note that $T : \mathrm{dom}(A_0) \to \mathcal{X}_0$ is a bounded operator. Then $T \in B_p(\mathrm{dom}(A_0), \mathcal{X}_0)$ if and only if $TA_0^{-1} \in B_p(\mathcal{X}_0)$.

Lemma A.3. *Assume that* $A_0 : \mathrm{dom}(A_0) \subseteq \mathcal{X}_0 \to \mathcal{X}_0$ *and* $\Gamma : \mathrm{dom}(\Gamma) \subseteq \mathcal{X}_0 \to \mathcal{X}_0$ *are closed linear operators on a Hilbert space* $\mathcal{X}_0$ *with domain* $\mathrm{dom}(A_0)$ *satisfying the following properties:*
(a) $\sigma(A_0)$ *and* $\sigma(\Gamma)$ *consist of discrete semi–simple eigenvalues,* $\{\alpha_n\}_{n\in\mathbb{Z}_+}$ *and* $\{\gamma_n\}_{n\in\mathbb{Z}_+}$*, respectively;*
(b) there is an orthonormal basis $\{e_n : n \in \mathbb{Z}_+\}$ *in* $\mathcal{X}_0$ *such that* $A_0 e_n = \alpha_n e_n$ *and* $\Gamma e_n = \gamma_n e_n$ *for all* $n \in \mathbb{Z}_+$*;*
There exists a constant $c > 0$ *such that*
(c) Γ *and* $A_0 + \frac{c^2}{4}\Gamma$ *are invertible;*
(d) $A_0 + ci\xi + \xi^2\Gamma^{-1}$ *is invertible for all* $\xi \in \mathbb{R}$*;*
(e) $\lim_{n\to\infty} |\frac{\alpha_n}{\gamma_n}| = \infty$*.*

Let $\mathcal{X} = \mathrm{dom}(|A_0\Gamma^{-1}|^{1/2}) \times \mathcal{X}_0$*. Then the operator* $A : \mathrm{dom}(A) \subseteq \mathcal{X} \to \mathcal{X}$ *defined by* $\mathrm{dom}(A) = \mathrm{dom}(A_0) \times \mathrm{dom}(|A_0\Gamma|^{1/2})$ *and* $A = \left[\begin{smallmatrix} 0 & \Gamma \\ A_0 & c\Gamma \end{smallmatrix}\right]$ *is the generator of a stable bi-semigroup. Moreover, the generator* A *has the diagonal form described in* (2.4)*, in particular it has an orthonormal basis of eigenvectors.*

Proof. Introduce the operator $(c^2\Gamma^2 + 4A_0\Gamma)^{1/2}$ by letting

$$(c^2\Gamma^2 + 4A_0\Gamma)^{1/2}e_n = (c^2\gamma_n^2 + 4\alpha_n\gamma_n)^{1/2}e_n, \tag{A.2}$$

where we fix the branch of square root so that $\sqrt{1} = 1$. Thus, the domain of the operator $(c^2\Gamma^2 + 4A_0\Gamma)^{1/2}$ is given by $\mathrm{dom}(|A_0\Gamma|^{1/2})$. We define on $\mathcal{X}_0$ the unbounded operators A_1, A_2 by $\mathrm{dom}(A_1) = \mathrm{dom}(A_2) = \mathrm{dom}(|A_0\Gamma|^{1/2})$ and

$$A_1 = \frac{c}{2}\Gamma + \frac{1}{2}(c^2\Gamma^2 + 4A_0\Gamma)^{1/2}, \tag{A.3}$$

$$A_2 = \frac{c}{2}\Gamma - \frac{1}{2}(c^2\Gamma^2 + 4A_0\Gamma)^{1/2}. \tag{A.4}$$

We remark that $A_1 - A_2 = (c^2\Gamma^2 + 4A_0\Gamma)^{1/2}$ is boundedly invertible on $\mathcal{X}_0$ by (c). Moreover,

$$\begin{aligned} &(A_1 - A_2)^{-1} : \mathcal{X}_0 \to \mathrm{dom}(|A_0\Gamma|^{1/2}) \\ &(A_1 - A_2)^{-1}\Big[\mathrm{dom}(|A_0\Gamma|^{1/2})\Big] = \mathrm{dom}(A_0\Gamma). \end{aligned} \tag{A.5}$$

Define the operator $\Psi : \mathcal{X}_0 \times \mathcal{X}_0 \to X = \mathrm{dom}(|A_0\Gamma^{-1}|^{1/2}) \times X_0$ by

$$\Psi = \begin{bmatrix} \Gamma & \Gamma \\ A_1 & A_2 \end{bmatrix} (A_1 - A_2)^{-1}. \tag{A.6}$$

Since

$$\mathrm{im}(\Gamma(A_1 - A_2)^{-1}) = \Gamma\Big[\mathrm{dom}(|A_0\Gamma|^{1/2})\Big] = \mathrm{dom}(|A_0\Gamma^{-1}|^{1/2})$$

we have that Ψ is well-defined and bounded. In addition, one can show that Ψ is boundedly invertible and

$$\Psi^{-1} = \begin{bmatrix} -\Gamma^{-1}A_2 & I \\ \Gamma^{-1}A_1 & -I \end{bmatrix} : \mathrm{dom}(|A_0\Gamma^{-1}|^{1/2}) \times \mathcal{X}_0 \to \mathcal{X}_0 \times \mathcal{X}_0.$$

From (A.5) and since

$$\begin{aligned} &\Gamma\Big[\mathrm{dom}(A_0\Gamma)\Big] = \mathrm{dom}(A_0) \\ &A_{1,2}\Big[\mathrm{dom}(A_0\Gamma)\Big] = \mathrm{dom}(|A_0\Gamma|^{1/2}). \end{aligned} \tag{A.7}$$

we infer that the following maps are also bounded:

$$\Psi : \mathrm{dom}(|A_0\Gamma|^{1/2}) \times \mathrm{dom}(|A_0\Gamma|^{1/2}) \to \mathrm{dom}(A_0) \times \mathrm{dom}(|A_0\Gamma|^{1/2}), \tag{A.8}$$

$$\Psi^{-1} : \mathrm{dom}(A_0) \times \mathrm{dom}(|A_0\Gamma|^{1/2}) \to \mathrm{dom}(|A_0\Gamma|^{1/2}) \times \mathrm{dom}(|A_0\Gamma|^{1/2}). \tag{A.9}$$

Finally, we introduce the operator

$$\tilde{A} = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} : \mathrm{dom}(\tilde{A}) := \mathrm{dom}(|A_0|^{1/2}) \times \mathrm{dom}(|A_0|^{1/2}) \to \mathcal{X}_0 \times \mathcal{X}_0.$$

Using (A.8) and (A.9) we calculate that $\Psi^{-1}A\Psi = \tilde{A}$. Moreover, from (d) we obtain that $\sigma(A_{1,2}) \cap i\mathbb{R} = \emptyset$ and since $\tilde{A}$ is an operator of multiplication we obtain that it generates a stable bi-semigroup. To finish the proof of lemma, we need to show that A has a diagonal structure. First, from (A.2) we infer that A_1 and A_2 have a diagonal structure. Using the canonical basis $\{e_n : n \in \mathbb{Z}_+\}$, we note that Φ is the operator of multiplication by the sequence

$$\begin{bmatrix} \gamma_n & \gamma_n \\ \frac{c}{2}\gamma_n(c^2\gamma_n^2 + 4\alpha_n\gamma_n)^{-1/2} + \frac{1}{2} & \frac{c}{2}\gamma_n(c^2\gamma_n^2 + 4\alpha_n\gamma_n)^{-1/2} - \frac{1}{2} \end{bmatrix}, \quad n \in \mathbb{Z}_+.$$

Since this matrix is diagonalizable, the lemma is proved. □

Appendix B. Compactness Results

In this appendix we formulate and prove several technical lemmas used in the proof of Theorem 3.9.

Lemma B.1. *Let $\mathcal{X}$ be a separable Hilbert space and assume $R \in \mathcal{B}_2(\mathcal{X})$. Then, the linear operator $\mathcal{R} : \ell^\infty(\mathbb{Z}_+, \mathcal{X}) \to \ell^\infty(\mathbb{Z}_+, \mathcal{X})$ defined by $\mathcal{R}(z_j)_{j\in\mathbb{Z}_+} = (Q_j R z_j)_{j\in\mathbb{Z}_+}$ is compact.*

Proof. We start by introducing the linear operators $\mathcal{R}_m : \ell^\infty(\mathbb{Z}_+, \mathcal{X}) \to \ell^\infty(\mathbb{Z}_+, \mathcal{X})$ defined by

$$\mathcal{R}(z_j)_{j\in\mathbb{Z}_+} = \big(Q_1 R z_1, \dots, Q_m R z_m, 0, \dots\big). \tag{B.1}$$

Using the definition above one can easily check that for any $\mathbf{z} = (z_j)_{j\in\mathbb{Z}_+}$ we have

$$\begin{aligned}\|(\mathcal{R}_m - \mathcal{R})\mathbf{z}\|_\infty^2 &= \sup_{k\geq m+1} \|Q_k R z_k\|^2 = \sup_{k\geq m+1} \|\sum_{n\in N_k} \langle R z_k, v_n\rangle v_n\|^2 \\ &= \sup_{k\geq m+1} \sum_{n\in N_k} |\langle R z_k, v_n\rangle|^2 = \sup_{k\geq m+1} \sum_{n\in N_k} |\langle z_k, R^* v_n\rangle|^2 \\ &\leq \sup_{k\geq m+1} \sum_{n\in N_k} \|z_k\|^2 \|R^* v_n\|^2 \leq \|\mathbf{z}\|_\infty \big(\sup_{k\geq m+1} \sum_{n\in N_k} \|R^* v_n\|^2\big).\end{aligned}$$

It follows that

$$\|\mathcal{R}_m - \mathcal{R}\|^2 \leq \sup_{k\geq m+1} \sum_{n\in N_k} \|R^* v_n\|^2 \leq \sum_{k\geq m+1} \sum_{n\in N_k} \|R^* v_n\|^2 = \sum_{n\in M_k} \|R^* v_n\|^2, \tag{B.2}$$

where $M_m = \bigcup_{k=m+1}^\infty N_k$ for $m \in \mathbb{Z}_+$. Since $R \in \mathcal{B}_2(\mathcal{X})$ we have that $R^* \in \mathcal{B}_2(\mathcal{X})$, and so

$$\lim_{m\to\infty} \sum_{n\in M_k} \|R^* v_n\|^2 = 0. \tag{B.3}$$

From (B.2)and (B.3) we conclude that $\mathcal{R}_m \to \mathcal{R}$ as $m \to \infty$ in $\mathcal{B}(\ell^\infty(\mathbb{Z}_+, \mathcal{X}))$. Therefore, since the linear operator $\mathcal{R}_m$ has finite rank for any $m \in \mathbb{Z}_+$, we infer that $\mathcal{R}$ is compact on $\ell^\infty(\mathbb{Z}_+, \mathcal{X})$, proving the lemma. □

In what follows we are going to prove that the linear operator $I - \mathcal{G}^+$ is Fredholm with index 0, by using that $\mathcal{G}^+$ defined in (3.31) has a representation of the form "small+compact". First, we note that for any $n \in \mathbb{Z}_+$, we have the following representation

$$\begin{aligned}B(s) = \varphi_n(s)B(s) + (1-\varphi_n(s))B(s) &= \mathcal{Q}^{(n)}\varphi_n(s)B(s) + (I - \mathcal{Q}^{(n)})\varphi_n(s)B(s) \\ &\quad + (1-\varphi_n(s))B(s).\end{aligned} \tag{B.4}$$

From (B.4) and (3.4) we obtain the following decomposition

$$G_j^+ = \sum_{\ell=0}^{9} G_{j,n}^{+,\ell}, \quad \text{for any} \quad n \in \mathbb{Z}_+, \tag{B.5}$$

where the linear operators $G_{j,n}^{+,\ell} : C_\mathrm{b}(\mathbb{R}_+, \mathcal{X}) \to C_\mathrm{b}(\mathbb{R}_+, \mathcal{X})$ are defined by

$$(G_{j,n}^{+,1} z)(t) = -\int_t^\infty e^{\varkappa_j(s-t)} T_+(t-s)\Big(\sum_{k=j}^{-1} Q_k\Big)\mathcal{Q}^{(n)}\varphi_n(s)B(s)z(s)\,\mathrm{d}s;$$

$$(G_{j,n}^{+,2} z)(t) = -\int_t^\infty e^{\varkappa_j(s-t)} T_-(s-t)\mathcal{Q}_-^{(n)}\varphi_n(s)B(s)z(s)\,\mathrm{d}s;$$

$$(G_{j,n}^{+,3}z)(t) = -\int_t^\infty e^{\varkappa_j(s-t)}T_+(t-s)(\sum_{k=j}^{-1}Q_k)(I-\mathcal{Q}^{(n)})\varphi_n(s)B(s)z(s)\,\mathrm{d}s;$$
$$(G_{j,n}^{+,4}z)(t) = -\int_t^\infty e^{\varkappa_j(s-t)}T_-(s-t)(\mathcal{Q}_- - \mathcal{Q}_-^{(n)})\varphi_n(s)B(s)z(s)\,\mathrm{d}s;$$
$$(G_{j,n}^{+,5}z)(t) = -\int_t^\infty e^{\varkappa_j(s-t)}T_+(t-s)(\sum_{k=j}^{-1}Q_k)(1-\varphi_n(s))B(s)z(s)\,\mathrm{d}s;$$
$$(G_{j,n}^{+,6}z)(t) = -\int_t^\infty e^{\varkappa_j(s-t)}T_-(s-t)\mathcal{Q}_-(1-\varphi_n(s))B(s)z(s)\,\mathrm{d}s;$$
$$(G_{j,n}^{+,7}z)(t) = \int_0^t \Gamma_j^+(t-s)\mathcal{Q}^{(n)}\varphi_n(s)B(s)z(s)\,\mathrm{d}s;$$
$$(G_{j,n}^{+,8}z)(t) = \int_0^t \Gamma_j^+(t-s)(I-\mathcal{Q}^{(n)})\varphi_n(s)B(s)z(s)\,\mathrm{d}s;$$
$$(G_{j,n}^{+,9}z)(t) = \int_0^t \Gamma_j^+(t-s)(1-\varphi_n(s))B(s)z(s)\,\mathrm{d}s. \tag{B.6}$$

From (3.30) and (B.5) it follows that

$$\mathcal{G}^+ = \sum_{\ell=0}^{9}\mathcal{G}_n^{+,\ell}, \quad \text{for any} \quad n\in\mathbb{Z}_+, \tag{B.7}$$

where the linear operators $\mathcal{G}_n^{+,\ell} : \ell^\infty(\mathbb{Z}_-, C_{\mathrm{b}}(\mathbb{R}_+,\mathcal{X})) \to \ell^\infty(\mathbb{Z}_-, C_{\mathrm{b}}(\mathbb{R}_+,\mathcal{X}))$ are defined by

$$\mathcal{G}_n^{+,\ell}(z_j)_{j\in\mathbb{Z}_-} = (G_{j,n}^{+,\ell}z_j)_{j\in\mathbb{Z}_-}. \tag{B.8}$$

In the next lemmas we will establish various estimates satisfied by the operators $\mathcal{G}_n^{+,\ell}$ and we will study their compactness properties.

Lemma B.2. *Assume* (2.1)–(2.2) *and Hypothesis 3.8. Then, the following estimate holds true:*

$$\|\mathcal{G}_n^{+,5} + \mathcal{G}_n^{+,6} + \mathcal{G}_n^{+,9}\|_{\mathcal{B}(\ell^\infty(\mathbb{Z}_-, C_{\mathrm{b}}(\mathbb{R}_+,\mathcal{X})))} \le \int_n^\infty \|B(s)\|\,\mathrm{d}s. \tag{B.9}$$

Proof. To begin the proof, we introduce the operator valued function $H_j^+ : \mathbb{R}^2 \to \mathcal{B}(\mathcal{X})$ by

$$H_j^+(t,s) = \begin{cases} \Gamma_j^+(t-s), & t\ge s,\\ -\Lambda_j^+(s-t), & t<s.\end{cases}$$

From (3.6) we conclude that

$$\|H_j^+(t,s)\| \le 1 \quad \text{for any} \quad (t,s)\in\mathbb{R}^2. \tag{B.10}$$

From (B.6) we infer that

$$\Big((G_{j,n}^{+,5} + G_{j,n}^{+,6} + G_{j,n}^{+,9})z\Big)(t) = \int_0^\infty (1-\varphi_n(s))H_j^+(t,s)B(s)z(s)\,\mathrm{d}s,$$

which implies that

$$\begin{aligned}\Big\|\Big((G_{j,n}^{+,5} + G_{j,n}^{+,6} + G_{j,n}^{+,9})z\Big)(t)\Big\| &\le \int_0^\infty (1-\varphi_n(s))\|H_j^+(t,s)\|\,\|B(s)\|\,\|z(s)\|\,\mathrm{d}s\\ &\le \Big(\int_0^\infty (1-\varphi_n(s))\|B(s)\|\,\mathrm{d}s\Big)\|z\|_{C_{\mathrm{b}}(\mathbb{R}_+,\mathcal{X})}\end{aligned}$$

$$\leq (\int_n^\infty \|B(s)\|\,\mathrm{d}s)\|z\|_{C_\mathrm{b}(\mathbb{R}_+,\mathcal{X})}$$

for any $t \geq 0$, $j \in \mathbb{Z}_-$ and $z \in C_\mathrm{b}(\mathbb{R}_+, \mathcal{X})$. Therefore, we obtain that

$$\|G_{j,n}^{+,5} + G_{j,n}^{+,6} + G_{j,n}^{+,9}\|_{\mathcal{B}(C_\mathrm{b}(\mathbb{R}_+,\mathcal{X})} \leq \int_n^\infty \|B(s)\|\,\mathrm{d}s.$$

The lemma follows shortly from the definition of the linear operators $\mathcal{G}_n^{+,\ell}$ given in (B.8). ☐

Lemma B.3. *Assume* (2.1)–(2.2) *and Hypothesis 3.8. Then, the following estimate holds true:*

$$\|\mathcal{G}_n^{+,4}\|_{\mathcal{B}(\ell^\infty(\mathbb{Z}_-,C_\mathrm{b}(\mathbb{R}_+,\mathcal{X})))} \leq \frac{1}{2\varkappa_{n+1}}\Big(\int_0^\infty \|B(s)\|^2\,\mathrm{d}s\Big)^{1/2}. \tag{B.11}$$

Proof. From (B.6) we know that

$$\begin{aligned}(G_{j,n}^{+,4}z)(t) &= -\int_t^\infty e^{\varkappa_j(s-t)}T_-(s-t)\Big(\sum_{k=n+1}^\infty Q_k\Big)\varphi_n(s)B(s)z(s)\,\mathrm{d}s\\ &= -\int_{\min\{t,n+1\}}^{n+1} e^{\varkappa_j(s-t)}\varphi_n(s)\sum_{k=n+1}^\infty\sum_{m\in N_k} e^{-a_m(s-t)}\langle B(s)z(s),v_m\rangle v_m\,\mathrm{d}s.\end{aligned}$$

Using the properties of the sequence $(\varkappa_j)_{j\in\mathbb{Z}\setminus\{0\}}$, we estimate

$$\begin{aligned}\|(G_{j,n}^{+,4}z)(t)\| &= \int_{\min\{t,n+1\}}^{n+1} e^{\varkappa_j(s-t)}\Big\|\sum_{k=n+1}^\infty\sum_{m\in N_k} e^{-a_m(s-t)}\langle B(s)z(s),v_m\rangle v_m\Big\|\,\mathrm{d}s\\ &\leq \int_{\min\{t,n+1\}}^{n+1} e^{\varkappa_j(s-t)}\Big(\sum_{k=n+1}^\infty\sum_{m\in N_k}\big|e^{-a_m(s-t)}\langle B(s)z(s),v_m\rangle\big|^2\Big)^{1/2}\mathrm{d}s\\ &= \int_{\min\{t,n+1\}}^{n+1} e^{\varkappa_j(s-t)}\Big(\sum_{k=n+1}^\infty e^{-2\varkappa_k(s-t)}\sum_{m\in N_k}\big|\langle B(s)z(s),v_m\rangle\big|^2\Big)^{1/2}\mathrm{d}s\\ &\leq \int_{\min\{t,n+1\}}^{n+1} e^{(\varkappa_j-\varkappa_{n+1})(s-t)}\Big(\sum_{k=n+1}^\infty\sum_{m\in N_k}\big|\langle B(s)z(s),v_m\rangle\big|^2\Big)^{1/2}\mathrm{d}s\\ &= \int_{\min\{t,n+1\}}^{n+1} e^{(\varkappa_j-\varkappa_{n+1})(s-t)}\|(\mathcal{Q}_- - \mathcal{Q}_-^{(n)})B(s)z(s)\|\,\mathrm{d}s\\ &\leq \Big(\int_{\min\{t,n+1\}}^{n+1} e^{2(\varkappa_j-\varkappa_{n+1})(s-t)}\,\mathrm{d}s\Big)^{\frac12}\Big(\int_{\min\{t,n+1\}}^{n+1}\|B(s)z(s)\|^2\,\mathrm{d}s\Big)^{\frac12}\end{aligned} \tag{B.12}$$

for any $t \geq 0$, $j \in \mathbb{Z}_-$ and $z \in C_\mathrm{b}(\mathbb{R}_+, \mathcal{X})$. Moreover, since $\varkappa_j < 0$ for all $j \in \mathbb{Z}_-$ one can readily check that

$$\int_{\min\{t,n+1\}}^{n+1} e^{2(\varkappa_j-\varkappa_{n+1})(s-t)}\,\mathrm{d}s \leq \int_{\min\{t,n+1\}}^{n+1} e^{-2\varkappa_{n+1}(s-t)}\,\mathrm{d}s \leq \frac{1}{2\varkappa_{n+1}} \tag{B.13}$$

for any $t \geq 0$, $j \in \mathbb{Z}_-$ and $n \in \mathbb{Z}_+$. From (B.12) and (B.13) we infer that

$$\|G_{j,n}^{+,4}z\|_{C_\mathrm{b}(\mathbb{R}_+,\mathcal{X})} \leq \frac{1}{2\varkappa_{n+1}}\Big(\int_0^\infty \|B(s)\|^2\,\mathrm{d}s\Big)^{1/2}\|z\|_{C_\mathrm{b}(\mathbb{R}_+,\mathcal{X})}$$

for any $j \in \mathbb{Z}_-$ and $z \in C_{\rm b}(\mathbb{R}_+, \mathcal{X})$. The lemma follows immediately from the definition of the linear operators $\mathcal{G}_n^{+,\ell}$ given in (B.8). □

Lemma B.4. *Assume* (2.1)–(2.2) *and Hypothesis 3.8. If* $m = \min\{j-1, -n-1\}$ *then the following estimate holds true:*

$$\|G_{j,n}^{+,8}\|_{\mathcal{B}(C_{\rm b}(\mathbb{R}_+,\mathcal{X}))} \leq \frac{1}{(2(\varkappa_j - \varkappa_m)^{\frac{1}{2}}} \Big(\int_0^\infty \|B(s)\|^2 \, {\rm d}s\Big)^{1/2}. \tag{B.14}$$

Proof. From (B.8) and the definition of Γ_j^+ in (3.4) we have that

$$\begin{aligned}
(G_{j,n}^{+,8} z)(t) &= \int_0^t \Gamma_j^+(t-s)(I - \mathcal{Q}^{(n)})\varphi_n(s)B(s)z(s)\,{\rm d}s \\
&= \int_0^{\min\{n+1,t\}} e^{-\varkappa_j(t-s)} T_+(t-s)(\sum_{k=-\infty}^{j-1} Q_k)(I - \mathcal{Q}^{(n)})\varphi_n(s)B(s)z(s)\,{\rm d}s \\
&= \int_0^{\min\{n+1,t\}} e^{-\varkappa_j(t-s)} T_+(t-s)(\sum_{k=-\infty}^{m} Q_k)\varphi_n(s)B(s)z(s)\,{\rm d}s.
\end{aligned}\tag{B.15}$$

Using (2.12) and (2.13) one can readily check that

$$\|T_+(t) \sum_{k=-\infty}^{j} Q_k\| = e^{\varkappa_j t} \quad \text{for any} \quad t \geq 0,\ j \in \mathbb{Z}_-. \tag{B.16}$$

From (B.16) and representation (B.15) we obtain that

$$\begin{aligned}
\|(G_{j,n}^{+,8} z)(t)\| &\leq \int_0^{\min\{n+1,t\}} e^{-\varkappa_j(t-s)} \|T_+(t-s)(\sum_{k=-\infty}^{m} Q_k)\| \, \|B(s)\| \, \|z(s)\| \,{\rm d}s \\
&\leq \Big(\int_0^{\min\{n+1,t\}} e^{(\varkappa_m - \varkappa_j)(t-s)} \|B(s)\| \,{\rm d}s\Big) \|z\|_{C_{\rm b}(\mathbb{R}_+,\mathcal{X})} \\
&\leq \Big(\int_0^{\min\{n+1,t\}} e^{(\varkappa_m - \varkappa_j)(t-s)} \,{\rm d}s\Big)^{\frac{1}{2}} \Big(\int_0^\infty \|B(s)\|^2 \,{\rm d}s\Big)^{\frac{1}{2}} \|z\|_{C_{\rm b}(\mathbb{R}_+,\mathcal{X})} \\
&\leq \frac{e^{(\varkappa_j - \varkappa_m)(\min\{n+1,t\}-t)}}{(2(\varkappa_j - \varkappa_m))^{\frac{1}{2}}} \Big(\int_0^\infty \|B(s)\|^2 \,{\rm d}s\Big)^{\frac{1}{2}} \|z\|_{C_{\rm b}(\mathbb{R}_+,\mathcal{X})}
\end{aligned}\tag{B.17}$$

for any $t \geq 0$, $j \in \mathbb{Z}_-$ and $z \in C_{\rm b}(\mathbb{R}_+, \mathcal{X})$. Since $m = \min\{j-1, -n-1\} \leq j-1 < j$ we have that $\varkappa_m < \varkappa_j$, which implies that

$$(\varkappa_j - \varkappa_m)(\min\{n+1, t\} - t) \leq 0 \quad \text{for any} \quad t \geq 0,\ j \in \mathbb{Z}_-,\ n \in \mathbb{Z}_+.$$

From (B.17) we conclude that

$$\|(G_{j,n}^{+,8} z)(t)\| \leq \frac{e^{(\varkappa_j - \varkappa_m)(\min\{n+1,t\}-t)}}{(2(\varkappa_j - \varkappa_m))^{\frac{1}{2}}} \Big(\int_0^\infty \|B(s)\|^2 \,{\rm d}s\Big)^{\frac{1}{2}} \|z\|_{C_{\rm b}(\mathbb{R}_+,\mathcal{X})}$$

$t \geq 0$, $j \in \mathbb{Z}_-$ and $z \in C_{\rm b}(\mathbb{R}_+, \mathcal{X})$, proving the lemma. □

Lemma B.5. *Assume* (2.1)–(2.2) *and Hypothesis 3.8. Then, the following estimate holds true:*

$$\|G_{j,n}^{+,2}\|_{\mathcal{B}(C_{\rm b}(\mathbb{R}_+,\mathcal{X}))} \leq \Big(-\frac{1}{2\varkappa_j}\Big)^{1/2} \Big(\int_0^\infty \|B(s)\|^2 \,{\rm d}s\Big)^{1/2}. \tag{B.18}$$

Proof. Since $|\varphi_n(s)| \leq 1$ and $\|T_-(s-t)\mathcal{Q}_-^{(n)}\| \leq 1$ for all $n \in \mathbb{Z}_+$ and $0 \leq t \leq s$, from (B.8) it follows that

$$\begin{aligned}
\|(G_{j,n}^{+,2}z)(t)\| &\leq \int_t^\infty e^{\varkappa_j(s-t)}\|T_-(s-t)\mathcal{Q}_-^{(n)}\|\,|\varphi_n(s)|\,\|B(s)\|^2\,\|z(s)\|\,\mathrm{d}s \\
&\leq \Big(\int_t^\infty e^{\varkappa_j(s-t)}\|B(s)\|\,\mathrm{d}s\Big)\|z\|_{C_{\mathrm{b}}(\mathbb{R}_+,\mathcal{X})} \\
&\leq \Big(\int_t^\infty e^{2\varkappa_j(s-t)}\mathrm{d}s\Big)^{\frac{1}{2}}\Big(\int_t^\infty \|B(s)\|\,\mathrm{d}s\Big)^{\frac{1}{2}}\|z\|_{C_{\mathrm{b}}(\mathbb{R}_+,\mathcal{X})} \\
&\leq \Big(-\frac{1}{2\varkappa_j}\Big)^{1/2}\Big(\int_0^\infty \|B(s)\|\,\mathrm{d}s\Big)^{1/2}\|z\|_{C_{\mathrm{b}}(\mathbb{R}_+,\mathcal{X})} \qquad \text{(B.19)}
\end{aligned}$$

for any $n \in \mathbb{Z}_+$, $t \geq 0$, $j \in \mathbb{Z}_-$ and $z \in C_{\mathrm{b}}(\mathbb{R}_+,\mathcal{X})$, proving the lemma. □

Lemma B.6. *Assume* (2.1)–(2.2) *and Hypothesis 3.8. For any $n \in \mathbb{Z}_+$ and $j \in \mathbb{Z}_-$ with $j \leq -n-1$, the following estimate holds true:*

$$\|G_{j,n}^{+,1}\|_{\mathcal{B}(C_{\mathrm{b}}(\mathbb{R}_+,\mathcal{X}))} \leq \frac{1}{2(\varkappa_{-n}-\varkappa_{-n-1})^{\frac{1}{2}}}\Big(\int_0^\infty \|B(s)\|^2\,\mathrm{d}s\Big)^{1/2}.$$

Proof. Assume $n \in \mathbb{Z}_+$, $j \in \mathbb{Z}_-$ and $j \leq -n-1$. Using the properties of the spectral projections Q_j's, from (B.8) it follows that

$$\begin{aligned}
(G_{j,n}^{+,1}z)(t) &= -\int_t^\infty e^{\varkappa_j(s-t)}T_+(t-s)\Big(\sum_{k=j}^{-1}Q_k\Big)\Big(\sum_{k=-n}^{n}Q_k\Big)\varphi_n(s)B(s)z(s)\,\mathrm{d}s \\
&= -\int_t^\infty e^{\varkappa_j(s-t)}T_+(t-s)\Big(\sum_{k=-n}^{-1}Q_k\Big)\varphi_n(s)B(s)z(s)\,\mathrm{d}s \qquad \text{(B.20)}
\end{aligned}$$

Since the linear subspace $\operatorname{im}\mathcal{Q}_-^{(n)}$ is *finite dimensional* and $T_+(t)(\sum_{k=-n}^{-1}Q_k)$ is diagonal on $\operatorname{im}\mathcal{Q}_-^{(n)}$, we infer that

$$\begin{aligned}
\|T_+(t)\Big(\sum_{k=-n}^{-1}Q_k\Big)\| &= \max\{|e^{-ta_m}| : m \in N_k, k=-n,\dots,-1\} \qquad \text{(B.21)} \\
&= \max\{e^{\varkappa_k t} : k=-n,\dots,-1\} = e^{\varkappa_{-n}t} \text{ for any } n \in \mathbb{Z}_+,\ t \leq 0.
\end{aligned}$$

Since $|\varphi_n(s)| \leq$ for all $n \in \mathbb{Z}_+$ and $s \in \mathbb{R}_+$, from (B.20) and (B.21)

$$\begin{aligned}
\|(G_{j,n}^{+,1}z)(t)\| &\leq \int_t^\infty e^{\varkappa_j(s-t)}\|T_+(t-s)\Big(\sum_{k=-n}^{-1}Q_k\Big)\|\,|\varphi_n(s)|\,\|B(s)\|\,\|z(s)\|\,\mathrm{d}s \\
&\leq \Big(\int_t^\infty e^{(\varkappa_j-\varkappa_{-n})(s-t)}\|B(s)\|\,\mathrm{d}s\Big)\|z\|_{C_{\mathrm{b}}(\mathbb{R}_+,\mathcal{X})} \\
&\leq \Big(\int_t^\infty e^{(\varkappa_{-n-1}-\varkappa_{-n})(s-t)}\|B(s)\|\,\mathrm{d}s\Big)\|z\|_{C_{\mathrm{b}}(\mathbb{R}_+,\mathcal{X})} \\
&\leq \Big(\int_t^\infty e^{2(\varkappa_{-n-1}-\varkappa_{-n})(s-t)}\,\mathrm{d}s\Big)^{\frac{1}{2}}\Big(\int_t^\infty \|B(s)\|^2\,\mathrm{d}s\Big)^{\frac{1}{2}}\|z\|_{C_{\mathrm{b}}(\mathbb{R}_+,\mathcal{X})} \\
&\leq \frac{1}{2(\varkappa_{-n}-\varkappa_{-n-1})^{\frac{1}{2}}}\Big(\int_0^\infty \|B(s)\|^2\,\mathrm{d}s\Big)^{\frac{1}{2}}\|z\|_{C_{\mathrm{b}}(\mathbb{R}_+,\mathcal{X})} \qquad \text{(B.22)}
\end{aligned}$$

for any $t \geq 0$ and $z \in C_{\mathrm{b}}(\mathbb{R}_+,\mathcal{X})$, proving the lemma. □

Next, we focus our attention on the compactness properties of the linear operators $\mathcal{G}_n^{+,\ell}$. To achieve this, we need to prove first some compactness results on $C_{\rm b}(\mathbb{R}_+,\mathcal{X})$.

Lemma B.7. *Assume Hypothesis 3.8 and let* $R:\mathbb{R}_+^2\to\mathcal{B}(\mathcal{X})$ *be a separable kernel, i.e.* $R(t,s)=R_1(t)R_2(s)$ *for all* $(t,s)\in\mathbb{R}_+^2$*, with* R_1,R_2 *strongly continuous on* $\mathbb{R}_+$*. Then, the following statements hold true:*

(i) *The linear operator* $G_n^R:C_{\rm b}(\mathbb{R}_+,\mathcal{X})\to C_{\rm b}(\mathbb{R}_+,\mathcal{X})$ *defined by* $(G_n^Rz)(t)=\int_t^\infty\varphi_n(s)R(t,s)B(s)z(s)\,{\rm d}s$*, is compact on* $C_{\rm b}(\mathbb{R}_+,\mathcal{X})$ *for each* $n\in\mathbb{Z}_+$*;*

(ii) *If, in addition,* $\sup_{t\geq 0}\|R_1(t)\|<\infty$*, the linear operator* $\tilde{G}_n^R:C_{\rm b}(\mathbb{R}_+,\mathcal{X})\to C_{\rm b}(\mathbb{R}_+,\mathcal{X})$ *defined by* $(\tilde{G}_n^Rz)(t)=\int_0^t\varphi_n(s)R(t,s)B(s)z(s)\,{\rm d}s$*, is compact on* $C_{\rm b}(\mathbb{R}_+,\mathcal{X})$ *for each* $n\in\mathbb{Z}_+$*.*

Proof. (i) Since $\varphi_n(t)=1$ for all $t\in[0,n]$ and $\varphi_n(t)=0$ for all $t\in[n+1,\infty)$ we have that

$$\begin{aligned}(G_n^Rz)(t)&=\int_{[t,\infty)\cap[0,n+1]}\varphi_n(s)R_1(t)R_2(s)B(s)z(s)\,{\rm d}s\\&=\varphi_{n+1}(t)R_1(t)\int_{\min\{t,n+1\}}^{n+1}\varphi_n(s)R_2(s)B(s)z(s)\,{\rm d}s\end{aligned}\tag{B.23}$$

From (B.23) we infer that the linear operator G_n^R can be decomposed as follows:

$$G_n^R=M_n^R\,S_n\,H_n^R\,J_n,\tag{B.24}$$

where M_n^R is the operator of multiplication by $\varphi_n(\cdot)R_1(\cdot)$ on $C_{\rm b}(\mathbb{R}_+,\mathcal{X})$, $H_n^R:C([0,n+1],\mathcal{X})\to C([0,n+1,\mathcal{X})$, $S_n:C([0,n+1],\mathcal{X})\to C_{\rm b}(\mathbb{R}_+,\mathcal{X})$, $J_n:C_{\rm b}(\mathbb{R}_+,\mathcal{X})\to C([0,n+1],\mathcal{X})$ are defined by

$$\begin{aligned}(H_n^Rw)(t)&=\int_t^{n+1}\varphi_n(s)R_2(s)B(s)w(s)\,{\rm d}s,\\(S_nw)(t)&=w(\min\{t,n+1\}),\ J_nz=z_{|[0,n+1]}.\end{aligned}\tag{B.25}$$

One can readily check that the linear operators S_n and J_n are bounded. Moreover, since $R_1(\cdot)$ is strongly continuous on $\mathbb{R}_+$, from the uniform boundedness principle we have that $\sup_{t\in[0,a]}\|R_1(t)\|<\infty$ for all $a>0$. In addition, since φ_{n+1} has compact support, it follows that $\sup_{t\geq 0}\varphi_{n+1}(t)\|R_1(t)\|<\infty$, which implies that M_n^R is bounded on $C_{\rm b}(\mathbb{R}_+,\mathcal{X})$. Since $R_2(s)$ is bounded on $\mathcal{X}$ and $B(s)$ is compact (actually, it is even Hilbert-Schmidt) on $\mathcal{X}$ for any $s\in[0,n+1]$, it follows that $R_2(s)B(s)$ is compact on $\mathcal{X}$ for any $s\in[0,n+1]$. Therefore, we obtain that H_n^R is compact on $C([0,n+1],\mathcal{X})$. From (B.24) and since the linear operators M_n^R,S_n,J_n are bounded we conclude that G_n^R is compact on $C_{\rm b}(\mathbb{R}_+,\mathcal{X})$, proving (i). The proof of (ii) follows closely that of (i). □

Lemma B.8. *Assume* (2.1)–(2.2) *and Hypothesis 3.8. Then, the linear operator* $G_{j,n}^{+,\ell}$ *is compact on* $C_{\rm b}(\mathbb{R}_+,\mathcal{X})$ *for all* $n\in\mathbb{Z}_+$*,* $j\in\mathbb{Z}_-$ *and* $\ell=1,2,3,7$*.*

Proof. This lemma is a direct consequence of the the previous lemma. Indeed, since $\operatorname{im}\sum_{k=j}^{-1}Q_k$ and $\operatorname{im}\mathcal{Q}_-^{(n)}$ are finite dimensional subspaces of $\mathcal{X}$ for all $n\in\mathbb{Z}_+$ and $j\in\mathbb{Z}_-$, we infer that the following kernels are *separable*:

$$R_{j,n}^{+,1}(t,s)=e^{\varkappa_j(s-t)}T_+(t-s)(\sum_{k=j}^{-1}Q_k)\mathcal{Q}^{(n)},\ R_{j,n}^{+,2}(t,s)=e^{\varkappa_j(s-t)}T_-(s-t)\mathcal{Q}_-^{(n)},$$

$$R^{+,3}_{j,n}(t,s) = e^{\varkappa_j(s-t)}T_+(t-s)(\sum_{k=j}^{-1} Q_k)(I - \mathcal{Q}^{(n)}). \tag{B.26}$$

From (B.8), (B.26) and Lemma B.7(i) we conclude that the operator $G^{+,\ell}_{j,n}$ is compact on $C_{\rm b}(\mathbb{R}_+,\mathcal{X})$ for all $n \in \mathbb{Z}_+$, $j \in \mathbb{Z}_-$ and $\ell = 1,2,3$. Similarly, since $\operatorname{im} \mathcal{Q}^{(n)}$ is a finite dimensional subspace of $\mathcal{X}$, from (3.4) it follows that the kernel

$$R^{+,7}_{j,n}(t,s) = \Gamma^+_j(t-s)\mathcal{Q}^{(n)} = e^{-\varkappa_j(t-s)}T_+(t-s)(\sum_{k=-\infty}^{j-1} Q_k)\mathcal{Q}^{(n)} \text{ is separable.} \tag{B.27}$$

Moreover, since $\Gamma^+_j(t-s)\mathcal{Q}^{(n)} = 0$ for all $n \in \mathbb{Z}_+$ and $j \in \mathbb{Z}_-$ with $j \le -n$, we have that $G^{+,7}_{j,n} = 0$ for all $n \in \mathbb{Z}_+$ and $j \in \mathbb{Z}_-$ with $j \le -n$. Therefore, we can assume without loss of generality that $-n+1 \le j \le -1$. From (B.27) we have that

$$R^{+,7}_{j,n}(t,s) = e^{-\varkappa_j(t-s)}T_+(t-s)(\sum_{k=-n}^{j-1} Q_k). \tag{B.28}$$

From (2.1)–(2.2) we have that

$$\begin{aligned}\|e^{-\varkappa_j t}T_+(t)(\sum_{k=-n}^{j-1} Q_k)\| &= \max\{|e^{-(\varkappa_j + a_m)}| : m \in N_k,\, k = -n,\dots,j-1\}\\ &= \max\{e^{(\varkappa_k - \varkappa_j)t} : k = -n,\dots,j-1\} \le 1 \text{ for all } t \ge 0.\end{aligned} \tag{B.29}$$

From (B.27) and (B.29) and Lemma B.7(ii) we conclude that $G^{+,7}_{j,n}$ is compact on $C_{\rm b}(\mathbb{R}_+,\mathcal{X})$, proving the lemma. □

In addition to the operators $\mathcal{G}^{+,\ell}_n$ defined in (B.8), we introduce the linear operators $\tilde{\mathcal{G}}^{+,\ell}_n : \ell^\infty(\mathbb{Z}_-, C_{\rm b}(\mathbb{R}_+,\mathcal{X})) \to \ell^\infty(\mathbb{Z}_-, C_{\rm b}(\mathbb{R}_+,\mathcal{X}))$, $\ell = 1,2,3$, as follows:

$$\begin{aligned}\tilde{\mathcal{G}}^{+,1}_n(z_j)_{j\in\mathbb{Z}_+} &= \big(G^{+,1}_{-1,n}z_{-1},\dots,G^{+,1}_{-n,n}z_{-n},0,\dots\big);\\ \tilde{\mathcal{G}}^{+,2}_n(z_j)_{j\in\mathbb{Z}_+} &= \big(G^{+,2}_{-1,n}z_{-1},\dots,G^{+,2}_{-n,n}z_{-n},0,\dots\big);\\ \tilde{\mathcal{G}}^{+,3}_n(z_j)_{j\in\mathbb{Z}_+} &= \big(0,\dots,0,\tilde{G}^{+,3}_{-n-1,n}z_{-n-1},\tilde{G}^{+,3}_{-n-2,n}z_{-n-2},\dots\big).\end{aligned} \tag{B.30}$$

Here the linear operators $\tilde{G}^{+,3}_{j,n} : C_{\rm b}(\mathbb{R}_+,\mathcal{X}) \to C_{\rm b}(\mathbb{R}_+,\mathcal{X})$ is defined by

$$(\tilde{G}^{+,3}_{j,n}z)(t) = -\int_t^\infty e^{\varkappa_j(s-t)}T_+(t-s)Q_j\varphi_n(s)B(s)z(s)\,\mathrm{d}s. \tag{B.31}$$

In the next lemma we prove an estimate that describes the difference between the operators $G^{+,3}_{j,n}$ and $\tilde{G}^{+,3}_{j,n}$.

Lemma B.9. *Assume* (2.1)–(2.2) *and Hypothesis 3.8. For any $n \in \mathbb{Z}_+$ and $j \in \mathbb{Z}_-$ with $j \le -n-2$, the following estimate holds true:*

$$\|G^{+,3}_{j,n} - \tilde{G}^{+,3}_{j,n}\|_{\mathcal{B}(C_{\rm b}(\mathbb{R}_+,\mathcal{X}))} \le \frac{1}{2(\varkappa_{j+1}-\varkappa_j)^{\frac12}}\Big(\int_0^\infty \|B(s)\|^2\,\mathrm{d}s\Big)^{1/2}. \tag{B.32}$$

Proof. First, we note that from (B.8) and (B.31) it follows that

$$\Big((G^{+,3}_{j,n} - \tilde{G}^{+,3}_{j,n})z\Big)(t) = -\int_t^\infty e^{\varkappa_j(s-t)}T_+(t-s)(\sum_{k=j+1}^{-n-1} Q_k)\varphi_n(s)B(s)z(s)\,\mathrm{d}s. \tag{B.33}$$

Since that the linear operator $T_+(t)(\sum_{k=j+1}^{-n-1} Q_k)$ is diagonal and $\mathrm{im}(\sum_{k=j+1}^{-n-1} Q_k)$ is a finite dimensional subspace of $\mathcal{X}$, we have that

$$\|T_+(t)(\sum_{k=j+1}^{-n-1} Q_k)\| = \max\{|e^{-ta_m}| : m \in N_k, k = j+1, \dots, -n-1\} \tag{B.34}$$
$$= \max\{e^{t\varkappa_k} : k = j+1, \dots, -n-1\} = e^{t\varkappa_{j+1}} \text{ for any } t \le 0.$$

Since $|\varphi_n(s)| \le$ for all $n \in \mathbb{Z}_+$ and $s \in \mathbb{R}_+$, from (B.33) and (B.34) we obtain that

$$\begin{aligned}\Big\|\Big((G^{+,3}_{j,n} - \tilde{G}^{+,3}_{j,n})z\Big)(t)\Big\| &\le \int_t^\infty e^{\varkappa_j(s-t)}\|T_+(t-s)(\sum_{k=j+1}^{-n-1} Q_k)\|\,\|B(s)\|\,\|z(s)\|\,\mathrm{d}s\\ &\le \Big(\int_t^\infty e^{(\varkappa_j-\varkappa_{j+1})(s-t)}\,\|B(s)\|\|z(s)\|\,\mathrm{d}s\Big)\\ &\le \Big(\int_t^\infty e^{2(\varkappa_j-\varkappa_{j+1})(s-t)}\,\mathrm{d}s\Big)^{\frac12}\Big(\int_t^\infty \|B(s)\|^2\|z(s)\|^2\,\mathrm{d}s\Big)^{\frac12}\\ &\le \frac{1}{2(\varkappa_{j+1}-\varkappa_j)^{\frac12}}\Big(\int_0^\infty \|B(s)\|^2\,\mathrm{d}s\Big)^{\frac12}\|z\|_{C_{\mathrm{b}}(\mathbb{R}_+,\mathcal{X})}\end{aligned} \tag{B.35}$$

for all $t \ge 0$ and $z \in C_{\mathrm{b}}(\mathbb{R}_+,\mathcal{X})$, proving the lemma. □

In the next lemma we are collecting some of the compactness results needed to prove the main result of this section.

Lemma B.10. *Assume* (2.1)–(2.2) *and Hypothesis 3.8. Then, the following statements hold true:*

(i) *The operators* $\tilde{\mathcal{G}}^{+,\ell}_n$, $\ell = 1,2,3$, *and* $\mathcal{G}^{+,7}_n$ *are compact on* $\ell^\infty(\mathbb{Z}_-, C_{\mathrm{b}}(\mathbb{R}_+,\mathcal{X}))$ *for all* $n \in \mathbb{Z}_+$*;*

(ii) $\|\mathcal{G}^{+,\ell}_n - \tilde{\mathcal{G}}^{+,\ell}_n\|_{\mathcal{B}(\ell^\infty(\mathbb{Z}_-,C_{\mathrm{b}}(\mathbb{R}_+,\mathcal{X})))} \to 0$ *as* $n \to \infty$ *for* $\ell = 1,2,3$.

Proof. (i) From Lemma B.8 and the definition of the linear operators $\tilde{\mathcal{G}}^{+,\ell}_n$ in (B.30) it follows immediately that $\tilde{\mathcal{G}}^{+,1}_n$ and $\tilde{\mathcal{G}}^{+,2}_n$ are compact on $\ell^\infty(\mathbb{Z}_-, C_{\mathrm{b}}(\mathbb{R}_+,\mathcal{X}))$ or all $n \in \mathbb{Z}_+$. Moreover, from (3.4) we have that $\Gamma^j_+(t-s)\mathcal{Q}^{(n)} = 0$, which implies that $G^{+,7}_{j,n} = 0$ for all $j \in \mathbb{Z}_-$ and $n \in \mathbb{Z}_+$ with $j \le -n$. Therefore, from Lemma B.8 we conclude that $\mathcal{G}^{+,7}_n$ is compact on $\ell^\infty(\mathbb{Z}_-, C_{\mathrm{b}}(\mathbb{R}_+,\mathcal{X}))$ for all $n \in \mathbb{Z}_+$.

Since $\varphi_n(s) = 0$ for all $s \in [n+1,\infty)$ from (B.31) we have that

$$\begin{aligned}(\tilde{G}^{+,3}_{j,n}z)(t) &= -e^{-\varkappa_j t}T_+(t)Q_j\int_{[t,\infty)\cap[0,n+1]} e^{\varkappa_j s}T_+(-s)Q_j\varphi_n(s)B(s)z(s)\,\mathrm{d}s\\ &= -e^{-\varkappa_j t}T_+(t)Q_j\int_{\min\{t,n+1\}}^{n+1} e^{\varkappa_j s}T_+(-s)Q_j\varphi_n(s)B(s)z(s)\,\mathrm{d}s\end{aligned} \tag{B.36}$$

for all $t \ge 0$, $z \in C_{\mathrm{b}}(\mathbb{R}_+,\mathcal{X})$, $n \in \mathbb{Z}_+$ and $j \in \mathbb{Z}_-$ with $j \le -n-1$. From (B.36) we conclude that the following decomposition holds true:

$$\tilde{\mathcal{G}}^{+,3}_n = \tilde{M}^{+,3}_n\,\tilde{S}_n\,\tilde{H}^{+,3}_n\,\tilde{J}_n, \tag{B.37}$$

where the linear operators $\tilde{M}^{+,3}_n : \ell^\infty(\mathbb{Z}_-, C_{\mathrm{b}}(\mathbb{R}_+,\mathcal{X})) \to \ell^\infty(\mathbb{Z}_-, C_{\mathrm{b}}(\mathbb{R}_+,\mathcal{X}))$, $\tilde{H}^{+,3}_n : C([0,n+1],\ell^\infty(\mathbb{Z}_-,\mathcal{X})) \to C([0,n+1,\ell^\infty(\mathbb{Z}_-,\mathcal{X}))$, are defined by

$$\tilde{M}^{+,3}_n(z_j)_{j\in\mathbb{Z}_-} = \Big(0,\dots,0,e^{-\varkappa_{-n}\cdot}T_+(\cdot)Q_{-n}, e^{-\varkappa_{-n-1}\cdot}T_+(\cdot)Q_{-n-1},\dots\Big);$$

$$(\tilde{H}_n^{+,3}\mathbf{w})(t) = \int_t^{n+1} \tilde{K}_n^{+,3}\mathbf{w}(s)\,\mathrm{d}s. \tag{B.38}$$

Here the function $\tilde{K}_n^{+,3} : \mathbb{R}_+ \to \mathcal{B}(\ell^\infty(\mathbb{Z}_-, \mathcal{X}))$ is given by

$$\tilde{K}_n^{+,3}(z_j)_{j\in\mathbb{Z}_-} = \Big(\varphi_n(s)e^{\varkappa_j s}T_+(-s)Q_jB(s)z_j\Big)_{j\in\mathbb{Z}_-} \tag{B.39}$$

Similar to (B.25) we introduce $\tilde{S}_n : C([0,n+1], \ell^\infty(\mathbb{Z}_-, \mathcal{X})) \to C_\mathrm{b}(\mathbb{R}_+, \ell^\infty(\mathbb{Z}_-, \mathcal{X}))$, $\tilde{J}_n : C_\mathrm{b}(\mathbb{R}_+, \ell^\infty(\mathbb{Z}_-, \mathcal{X})) \to C([0,n+1], \ell^\infty(\mathbb{Z}_-, \mathcal{X}))$ by

$$(\tilde{S}_n\mathbf{w})(t) = \mathbf{w}(\min\{t, n+1\}),\ \tilde{J}_n\mathbf{z} = \mathbf{z}_{|[0,n+1]}. \tag{B.40}$$

From (3.5) one can readily check that $\|T_(t)Q_j\| = e^{\varkappa_j t}$ for all $t \geq 0$ and $j \in \mathbb{Z}_-$, which implies that $M_n^{+,3}$ is a bounded linear operator on $\ell^\infty(\mathbb{Z}_-, C_\mathrm{b}(\mathbb{R}_+, \mathcal{X}))$. Furthermore, since it is straightforward to check that the linear operators $\tilde{S}_n$ and $\tilde{J}_n$ are also bounded, to finish the proof of lemma we only need to show that the linear operator $\tilde{H}_n^{+,3}$ is compact on $C([0,n+1], \ell^\infty(\mathbb{Z}_-, \mathcal{X}))$.

Using (3.5) again, we obtain that $\|e^{\varkappa_j s}T_+(-s)Q_j\| = 1$ for all $j \in \mathbb{Z}_-$ and $s \geq 0$. Moreover, since $T_+(-s)Q_j = Q_jT_+(-s)Q_j$ for all $j \in \mathbb{Z}_-$ and $s \geq 0$ and $B(s) \in \mathcal{B}_2(\mathcal{X})$ for all $s \geq 0$, from Lemma B.1 we infer that $\tilde{K}_n^{+,3}(s)$ is compact on $\ell^\infty(\mathbb{Z}_-, \mathcal{X})$ for all $s \geq 0$. From (B.38) it follows that $\tilde{H}_n^{+,3}$ is compact on $C([0,n+1], \ell^\infty(\mathbb{Z}_-, \mathcal{X}))$, proving (i).

Assertion (ii) follows from Lemma B.5, Lemma B.6 and Lemma B.9. □

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Department of Mathematics, University of Missouri, Columbia, MO 65211, USA
E-mail address: latushkiny@missouri.edu
URL: http://www.math.missouri.edu/personnel/faculty/latushkiny.html

Department of Mathematics, Indiana University, Bloomington, IN 47405, USA
E-mail address: apogan@indiana.edu
URL: http://mypage.iu.edu/~apogan/