QUASI-GRADIENT SYSTEMS, MODULATIONAL DICHOTOMIES, AND STABILITY OF SPATIALLY PERIODIC PATTERNS

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Abstract. Extending the approach of Grillakis–Shatah–Strauss, Bronski–Johnson–Kapitula, and others for Hamiltonian systems, we explore relations between the constrained variational problem

$$\min_{X:C(X)=c_0} \mathcal{E}(X), \quad c_0 \in \mathbb{R}^r,$$

and stability of solutions of a class of degenerate "quasi-gradient" systems $dX/dt = -M(X)\nabla\mathcal{E}(X)$ admitting constraints, including Cahn-Hilliard equations, one- and multi-dimensional viscoelasticity, and coupled conservation law-reaction diffusion systems arising in chemotaxis and related settings. Using the relation between variational stability and the signature of $\partial c/\partial\omega \in \mathbb{R}^{r\times r}$, where $c(\omega) = C(X^*_\omega) \in \mathbb{R}^r$ denote the values of the imposed constraints and $\omega \in \mathbb{R}^r$ the associated Lagrange multipliers at a critical point X^*_ω, we obtain as in the Hamiltonian case a general criterion for co-periodic stability of periodic waves, illuminating and extending a number of previous results obtained by direct Evans function techniques. More interestingly, comparing the form of the Jacobian arising in the co-periodic theory to Jacobians arising in the formal Whitham equations associated with modulation, we recover and substantially generalize a previously mysterious "modulational dichotomy" observed in special cases by Oh–Zumbrun and Howard, showing that co-periodic and sideband stability are incompatible. In particular, we both illuminate and extend to general viscosity/strain-gradient effects and multidimensional deformations the result of Oh-Zumbrun of universal modulational instability of periodic solutions of the equations of viscoelasticity with strain-gradient effects, considered as functions on the whole line. Likewise, we generalize to multi-dimensions corresponding results of Howard on periodic solutions of Cahn–Hilliard equations.

1. Introduction

In this paper, we investigate from a general point of view stability of standing spatially periodic solutions of a broad class of equations with gradient or "quasi-gradient" structure including Cahn–Hilliard, viscoelasticity, and coupled conservative–reaction diffusion equations. Our main goals are to (i) understand modulational stability of periodic viscoelastic solutions with physically relevant viscosity/strain-gradient coefficients and multi-dimensional deformations, about which up to now very little has been known, and (ii) understand through gradient or related structure the commonality of results that have emerged through direct Evans-function techniques in this and other types of systems.

In particular, we seek to better understand an apparently special modulational dichotomy that was established for a special model of one-dimensional viscoelasticity by Oh–Zumbrun in [44], but later shown by Howard [26] to hold also for the Cahn–Hilliard equation, asserting that, under

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rather general conditions, *co-periodic and sideband stability are mutually incompatible:* in particular, periodic solutions are always modulationally unstable. Both of these results were obtained by direct Evans-function computations, leaving the underlying mechanism unclear. Moreover, in the viscoelastic case, the technical condition of monotonicity of period with respect to amplitude left open the possibility that there might be special parameter regimes in which periodic waves could be stable. Further, the method of proof does not appear to generalize to multi-dimensional planar periodic waves, as studied for example in [58, 6].

Thus, these prior results left intriguing questions regarding the possibility of stable periodic viscoelastic solutions. We note that, even though second-order reaction-diffusion equations cannot, by Sturm–Liouville considerations, support stable periodic waves, nor, for the same reasons, can conservative scalar Cahn–Hilliard equations, the Swift-Hohenberg equations give a well-known example of a higher-order reaction-diffusion equation that does support stable waves [41, 53], and generalized Kuramoto-Sivashinsky equations are another well-known example showing that conservation laws can likewise support stable periodic waves; see [30, 4, 24]. Thus, the conservative structure of the equations of viscoelasticity is not a priori an obstruction to stability.

A clue noted in [44, 61] regarding possible common origins of the modulational dichotomies for Cahn–Hilliard and viscoelasticity equations are that both serve as models for phase-transitional elasticity, and both share a nonincreasing Cahn–Hilliard/van der Waals energy functional $\mathcal{E}$. Indeed, the Cahn–Hilliard equation is well-known to be the gradient system with respect to H^{-1} of the functional $\mathcal{E}$, that is,

$$(d/dt)X(t) = -M\nabla\mathcal{E}(X(t)), \tag{1.1}$$

where $M = (i\partial_x)(i\partial_x)^* = -\partial_x^2$ is symmetric positive semidefinite. It has been conjectured [44, 61] that the similar results obtained for viscoelasticity are due to this "hidden variational structure."

Here, we show that this is true, in an extremely strong sense. Namely, we show that the equations of viscoelasticity with strain-gradient effects may be put into the same form (1.1), but with M only *positive semidefinite* (i.e., $M + M^* \geq 0$) and not symmetric- a structure that might be called "generalized gradient" or "quasi-gradient" form. More, we show that, under an additional property (E1) of viscoelasticity, satisfied automatically in the gradient case $M = M^*$, all of the results that have so far been shown for this system, including the results on modulational dichotomy, may be deduced entirely from this quasi-variational structure, constrained variational stability considerations like those used in [18, 9] and elsewhere to study stability of Hamiltonian waves, and a novel connection observed between the Jacobian and Lagrangian multipliers arising in constrained variational theory and the Whitham modulation equations formally governing sideband behavior. The latter we regard as the main new contribution of this paper.

Indeed, we are able to put each of the mentioned applications into this framework and obtain a suite of similar results by these common methods. As in any investigation overlapping with classical calculus of variations and Sturm–Liouville theory, many of our results in the simpler cases considered here can be obtained by other means, and a great many have (no doubt more than the ones we have been able to cite). Still more of our results can be obtained alternatively in a slightly weakened form by direct Evans function computations, which give a sort of mod two version of Sturm–Liouville applying in more general circumstances; see for example, the discussion of Sections 4.1.1 and 4.5.

However, a significant number of our results appear also to be new, in particular (surprisingly) modulational instability of multiply periodic Cahn–Hilliard solutions (though see Remark 5.7 on the two-dimensional case), Theorem 5.6; modulational instability of singly and multiply periodic coupled conservation–reaction diffusion (in particular, Keller–Segel) solutions, Theorem 5.9; universal modulational instability, without the technical condition of monotonicity of period vs. amplitude, of one-dimensional periodic viscoelasticity waves, Theorem 5.1; and modulational instability of the branch of planar multi-dimensional deformation solutions of viscoelasticity connected by transversal homotopy to the decoupled, one-dimensional case, Corollary 5.2. By related techniques, we recover also versions of the results of [17, 61, 48] on instability of homoclinic "spike" type solutions on the whole line; see Section 6.2.

Also, though many of these ideas appear elsewhere in disparate sources, we feel that there is some value in putting a broad variety of problems into a common framework, continuing and extending the unification initiated in [47, 48]. In particular, we both formalize the heuristic relation pointed out in [48] between coupled conservative–reaction diffusion and Cahn–Hilliard equations and extend this to the quasi-gradient viscoelastic case. We discuss also the natural connection to the Hamiltonian case of systems $(d/dt)X(t) = J\nabla\mathcal{E}(X(t))$ with $J = -J^*$ skew symmetric. Here, interestingly, the appearance of conservation laws for generators $-MA$ with M noninvertible is paralleled by the appearance for J noninvertible of conservation laws associated with noninvertibility of J rather than Noetherian symmetries; see (3.8). Finally, in Section 5.5, we give a unified explanation of the previously-observed modulational dichotomies from the variational point of view, showing that the mechanisms lying behind these are unconstrained instability of A, together with a relation between Morse indices of the Floquet operators associated with MA and A, and the aforementioned property (E1) that the center subspace of MA consist of $\ker MA$.

1.1. **Discussion and open problems.** As concerns our primary interest of classifying stability of periodic viscoelastic patterns, we have resolved some of the main problems pointed out in [44, 58], but also pointed the way to some interesting further directions for investigation. In particular, it is an extremely interesting open question whether there could be a bifurcation to a stable branch of planar multideformational patterns as solutions are continued from the decoupled, effectively one-dimensional deformation, case. Another interesting open question is whether there are there other periodic viscoelastic solutions that are 'genuinely multi-dimensional" in the sense that they are not continuable from the decoupled case. These and related problems for homoclinic solutions are being investigated in [6] by numerical Evans function techniques.

We note that an efficient alternative Evans method is proposed here for self-adjoint problems, based on the Sturm-Liouville duality between spatial and frequency variables x and λ; see the last paragraph of Section 4.1.1. This also would be interesting to explore. Finally, it would be interesting to look for stable multiply periodic viscoelasticity solutions, perhaps corresponding to waves seen in buckling of viscoelastic materials [23, 3]); see Section 5.1.1.

Note. We have been informed after the completion of this work that our analysis in Section 4.1 overlaps with unpublished results of Bronski and Johnson on nonlocal Cahn–Hilliard equations.

2. Preliminaries: variational stability and Sylvester's law of inertia

We begin by recalling a fundamental relation ((2.12) below) underlying the Hamiltonian stability results of [18], [9], etc., between stability of constrained critical points and the Jacobian $\partial c/\partial\omega$ of the map from Lagrange multipliers to constraints. This principle seems to go far back, at least implicitly, in the physics (and particularly the thermodynamics) literature [10, 37]. For a general treatment from the Calculus of Variations point of view, see for example [39] and references therein.[4] Here, we give a particularly simple derivation[5] based on Sylvester's law of inertia.

[4] Curiously, the Jacobian $(\partial c/\partial\omega)(\omega^*)$ emphasized in [18] and later works is never explicitly mentioned in [39], even though the physical importance of the Lagrange multiplier is stressed. Rather, it appears in the equivalent form $-y^T A_{\omega^*} y = -\langle\nabla C, y\rangle$, $A_{\omega^*} y := \nabla C$, arising naturally in the analysis (and in practice most easily computable).

[5] Essentially a redaction of the argument of [39]

2.1. **Constrained minimization and Lagrange Multiplier formulation.** Consider the general constrained minimization problem

$$\min_{X:C(X)=c_0} \mathcal{E}(X), \quad C=(C_1,\dots,C_r): \mathbf{H}\to\mathbb{R}^r, \tag{2.1}$$

on a Hilbert space $\mathbf{H}$. Here, the constraint function C is C^2 on $\mathbf{H}$. Also, we assume that $\mathcal{E}$ is differentiable on $\mathrm{Diff}(\mathcal{E})$ a dense subspace of $\mathbf{H}$, and that for every $X\in\mathrm{Diff}(\mathcal{E})$, $\nabla\mathcal{E}(X)$ and $\nabla^2\mathcal{E}(X)$ are well-defined at least on a subset of $\mathrm{Diff}(\mathcal{E})$ that is dense in $\mathbf{H}$. In addition, we assume

(H1) ∇C is full rank in the vicinity under consideration.

Under assumption (H1), the feasible set $\{C(X)=c_0\}$ by the Implicit Function Theorem defines a local C^2 manifold of co-dimension r in the vicinity of any point X^* with $C(X^*)=c_0$, tangent to $\nabla C(X^*)^\perp$,[6] on which one may in principle carry out standard Calculus of Variations on local coordinate charts.

Alternatively, one may solve an optimization problem on an enlarged flat space, following the method of Lagrange. Introduce the associated Lagrange multiplier problem

$$\min_{X\in\mathbf{H},\,\omega\in\mathbb{R}^r} \mathcal{E}_\omega(X) \quad \text{where} \quad \mathcal{E}_\omega(X):=\mathcal{E}(X)+\omega\cdot(C(X)-c), \tag{2.2}$$

$\omega\in\mathbb{R}^r$ are Lagrange multipliers, and the constrained Hessian (in X only) is given by:

$$A_\omega := \nabla^2\mathcal{E}_\omega(X^*) = \nabla^2\mathcal{E}(X^*)+\omega\cdot\nabla^2 C(X^*). \tag{2.3}$$

Critical points $(X^*,\omega^*)\in\mathrm{Diff}(\mathcal{E})\times\mathbb{R}^r$ of (2.2) satisfy $C(X^*)=c_0$ and

$$\nabla\mathcal{E}(X^*) = -\sum_{j=1}^r \omega_j^*\nabla C_j(X^*). \tag{2.4}$$

We recall the following well-known characterizations of stability.

Proposition 2.1 (First derivative test)**.** *An element of $X^*\in\mathrm{Diff}(\mathcal{E})$ is a critical point of* (2.1) *if and only if there exists ω^* such that (X^*,ω^*) is a critical point of* (2.2) *for some $\omega^*\in\mathbb{R}^r$.*

Proof. Critical points of (2.1) satisfy $C(X^*)=c_0$ and $\langle\nabla\mathcal{E}(X^*),Y\rangle=0$ for $Y\in\nabla C(X^*)^\perp$, yielding $\nabla\mathcal{E}(X^*)\in\mathrm{Span}\{\nabla C_j(X^*):j=1,\dots,r\}$, or (2.4). □

Proposition 2.2 (Second derivative test)**.** *Assume hypothesis* (H1)*. Then the following assertions hold true:*

(i) *A critical point $X^*\in\mathrm{Diff}(\mathcal{E})$ is strictly stable in $\mathrm{Diff}(\mathcal{E})$ (a strict local minimum of* (2.1) *among competitors in $\mathrm{Diff}(\mathcal{E})$) if A_{ω^*} is positive definite on $\nabla C(X^*)^\perp\cap\mathrm{Diff}(\mathcal{E})$,*

(ii) *It is unstable in $\mathrm{Diff}(\mathcal{E})$, (not a nonstrict local minimizer of* (2.1) *among competitors in $\mathrm{Diff}(\mathcal{E})$), hence in $\mathbf{H}$, if A_{ω^*} is not positive semi-definite on $\nabla C(X^*)^\perp\cap\mathrm{Diff}(\mathcal{E})$,*

Proof. Equivalently, $(d^2\mathcal{E}(X(t))/dt^2)|_{t=0}<0$ (resp. $\not\leq 0$) for any path $X(t)\in\mathrm{Diff}(\mathcal{E})$ with $X(0)=X^*$, $C(X(t))\equiv c_0$. Taylor expanding, we have that $y(t):=X(t)-X_*=tY_1+t^2Y_2+\mathcal{O}(t^3)$, where $Y_1\in\nabla C(X^*)^\perp$ and $\langle\nabla C_j(X^*),Y_2\rangle=-\frac{1}{2}\langle Y_1,\nabla^2C_j(X^*)Y_1\rangle$ for any $j=1,\dots,r$. Using, (2.4) we obtain that

$$\begin{aligned}\frac{1}{2}(d^2\mathcal{E}(X(t))/dt^2)|_{t=0} &= \langle Y_1,\frac{1}{2}\nabla^2\mathcal{E}(X^*)Y_1\rangle+\langle\nabla\mathcal{E}(X^*),Y_2\rangle\\ &= \langle Y_1,\frac{1}{2}\nabla^2\mathcal{E}(X^*)Y_1\rangle-\sum_{j=1}^r\omega_j^*\ \langle\nabla C_j(X^*),Y_2\rangle\end{aligned}$$

[6] We use this notation to denote $\{\nabla C_j(X^*):j=1,\dots,r\}^\perp$, slightly abusing the notation.

$$= \langle Y_1, \frac{1}{2}\nabla^2\mathcal{E}(X^*)Y_1\rangle - \frac{1}{2}\sum_{j=1}^{r}\omega_j^* \langle Y_1, \nabla^2 C_j(X^*)Y_1\rangle = \frac{1}{2}\langle Y_1, A_{\omega^*}Y_1\rangle,$$

from which both assertions readily follow. □

Remark 2.3. Note that the second variation conditions for the Lagrange multiplier problem are with respect to the constrained Hessian, and along the tangent feasible space $\nabla C(X^*)^\perp$ only.

Remark 2.4. The conclusions of this section are strongest, evidently, when $\mathrm{Diff}(\mathcal{E}) = \mathbf{H}$, in which case strict stability implies variation in energy controls variation in $\mathbf{H}$, or

$$\|X_1 - X^*\|^2_{\mathbf{H}} \lesssim \mathcal{E}(X_1) - \mathcal{E}(X^*) \text{ for } \|X_1 - X^*\|_{\mathbf{H}} \text{ sufficiently small.} \tag{2.5}$$

This holds, for example, for the Cahn–Hilliard and generalized KdV energies considered in Sections 4.1 and 4.4. However, the Keller–Segel and viscoelastic energies considered in Sections 4.2 and 4.3 are not even continuous on $\mathbf{H}$, hence the more general phrasing here; see Remarks 4.13 and 4.12.

2.2. **Sylvester's Law and relative signature.** Let $Q : \mathrm{dom}(Q) \subset \mathbf{H} \to \mathbf{H}$ be a self-adjoint operator on $\mathbf{H}$. For $\alpha \in \{-, 0, +\}$ we denote by $\mathcal{S}_\alpha(Q)$ the set of all nontrivial *finite* dimensional subspaces $V \subset \mathrm{dom}(Q)$ such that $Q_{|V}$ is negative definite, identically zero, and positive definite, respectively. Define[7]

$$\sigma_\alpha(Q) = \begin{cases} \sup\{\dim V : V \in \mathcal{S}_\alpha(Q)\}, & \text{if } \mathcal{S}_\alpha(Q) \neq \emptyset \\ 0, & \text{if } \mathcal{S}_\alpha(Q) = \emptyset \end{cases} \in \mathbb{Z}_+ \cup \{\infty\}, \tag{2.6}$$

$\alpha \in \{-, 0, +\}$. We introduce the signature of the self-adjoint operator Q by

$$\sigma(Q) := (\sigma_-(Q), \sigma_0(Q), \sigma_+(Q)). \tag{2.7}$$

Recall now the following fundamental observation.

Proposition 2.5 (Sylvester's Law of Inertia)**.** *The signature σ is invariant under coordinate changes, i.e., $\sigma(Q) = \sigma(S^*QS)$ for any bounded invertible coordinate change S.*

Proof. Invariance follows immediately since the change of variables S is invertible and $V \in \mathcal{S}_\alpha(Q)$ implies that $S^{-1}(V) \in \mathcal{S}_\alpha(S^*QS)$ and $\dim V = \dim S^{-1}(V)$; $U \in \mathcal{S}_\alpha(S^*QS)$ implies that $S(U) \in \mathcal{S}_\alpha(Q)$ and $\dim U = \dim S(U)$. □

In the sequel, we will need the following proposition:

Proposition 2.6. *Assume $Q : \mathrm{dom}(Q) \subset \mathbf{H} \to \mathbf{H}$ is a self-adjoint operator. The the following assertions hold true:*

(i) *If B is a self-adjoint, invertible, $s \times s$ matrix and $R : \mathrm{dom}(Q) \otimes \mathbb{C}^s \to \mathbf{H} \otimes \mathbb{C}^s$ is defined by $R = \begin{pmatrix} Q & 0 \\ 0 & B \end{pmatrix}$, then*

$$\sigma_\alpha(R) = \sigma_\alpha(Q) + \sigma_\alpha(B), \quad \alpha \in \{-, 0, +\}; \tag{2.8}$$

(ii) *If $\sigma_\alpha(Q)$, $\alpha \in \{-, 0, +\}$, is finite then it is equal to the dimensions of the stable, zero, and unstable eigenspaces of Q, respectively.*

Proof. (i) First we note that our assertion is true if B is a nonzero real number. By induction, we conclude that (2.8) is also true provided the matrix B is a diagonal matrix with nonzero real entries.

[7]Note that in the case of σ_0, we require Q, not the quadratic form $\langle \cdot, Q\cdot\rangle$, to be identically zero on V so that $\sigma_0(Q)$ is the dimension of the kernel of Q.

Next, we look at the general case when B is a general self-adjoint matrix. Then, there exists P_B an orthogonal matrix and D a diagonal matrix such that $B = P_B^T D P_B$. Since any orthogonal matrix is unitary we have that $P_B^T = P_B^*$, $\begin{pmatrix} I_{\mathbf{H}} & 0 \\ 0 & P_B \end{pmatrix}$ is invertible and

$$R = \begin{pmatrix} I_{\mathbf{H}} & 0 \\ 0 & P_B \end{pmatrix}^* \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} I_{\mathbf{H}} & 0 \\ 0 & P_B \end{pmatrix}.$$

Assertion (i) follows from Proposition 2.5 and the observation that (2.8) is true for diagonal matrices B with nonzero real entries.

We note that assertion (ii) is trivial for $\alpha = 0$. Assume that $\sigma_+(Q)$ is finite. Then, there exists a subspace $\tilde{\mathbf{H}}$, a self-adjoint, negative-definite operator $\tilde{Q} : \mathrm{dom}(\tilde{Q}) \subset \tilde{\mathbf{H}} \to \tilde{\mathbf{H}}$ and D a diagonal $s \times s$ matrix with positive real entries, where s is equal to the dimension of the unstable space of Q, so that $Q = \begin{pmatrix} \tilde{Q} & 0 \\ 0 & D \end{pmatrix}$. Since $\sigma_+(\tilde{Q}) = 0$ and $\sigma_+(D) = s$, our conclusion follows immediately from (i). $\square$

Assume now

(H2) Near (X^*, ω^*), there is a C^1 family $(X, c)(\omega)$ of critical points of (2.2).

This follows, for example, by the Implicit Function Theorem,

applied to (2.4) if $\ker A_{\omega^*}$ is trivial and also A_{ω^*} is uniformly invertible. Another situation in which (H2) follows, of particular interest in the applications considered here, is when $\ker A_{\omega^*}$ is due to group symmetries of the problem leaving both energy $\mathcal{E}$ and constraints C invariant. and also $A_{\omega^*}|_{(\ker A_{\omega^*})^\perp}$ is uniformly invertible, i.e., A_{ω^*} has a spectral gap. In general, it may be obtained as part of an existence theory for Euler-Lagrange equation (2.4) carried out by other means. We note that (H2) in any case implies also

(2.9) $$\ker A_{\omega^*} \perp \nabla C(X^*),$$

(by Fredholm alternative), as holds automatically when $\ker A_{\omega_*}$ is generated by group invariances.

Assume also

(H3) The matrix $(\partial c/\partial \omega)(\omega^*)$ is invertible.

Differentiating the Euler–Lagrange relation (2.4) with respect to ω, we obtain the key relation

(2.10) $$A_{\omega^*}(\partial X/\partial \omega_j)(\omega^*) = -\nabla C_j(X^*) \quad \text{for all} \quad j = 1, \ldots, r$$

and its consequence

(2.11) $$(\partial c_k/\partial \omega_j)(\omega^*) = -\langle (\partial X/\partial \omega_j)(\omega^*), A_{\omega^*}(\partial X/\partial \omega_k)(\omega^*)\rangle$$

for all $j, k = 1, \ldots, r$, from which we see that $(\partial c/\partial \omega)(\omega^*)$ is a symmetric form. Our main result is the following identity relating the signatures of $A_{\omega^*}|_{\nabla C(X^*)^\perp}$, A_{ω^*}, and $(\partial c/\partial \omega)(\omega^*)$.

Theorem 2.7. *Assuming* (H1)–(H3),

(2.12) $$\sigma(A_{\omega^*}) = \sigma\Big(A_{\omega^*}|_{\nabla C(X^*)^\perp}\Big) + \sigma\Big(-\partial c/\partial \omega)(\omega^*)\Big).$$

Proof. To prove the theorem we use Sylvester's Law. First, we note that from (H3), (2.10) and (2.11) it follows that

(2.13) $$\mathbf{H} = \nabla C(X^*)^\perp \oplus \mathcal{Z}_{\omega^*},$$

where $\mathcal{Z}_{\omega^*} = \text{Span}\{(\partial X/\partial\omega_j)(\omega^*) : j = 1,\ldots,r\}$. Next, we denote by $\Pi_{\nabla C(X^*)^\perp}$ and $\Pi_{\mathcal{Z}_{\omega^*}}$ the orthogonal projections onto $\nabla C(X^*)^\perp$ and $\mathcal{Z}_{\omega^*}$, respectively. Using (2.13) we infer that $S : \nabla C(X^*)^\perp \times \mathcal{Z}_{\omega^*} \to \mathbf{H}$ defined by

$$S\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = h_1 + h_2 \tag{2.14}$$

is bounded, invertible with bounded inverse. Moreover, from (2.10) and (2.11) we obtain that

$$\Pi_{\nabla C(X^*)^\perp} A_{\omega^*} \Pi_{\mathcal{Z}_{\omega^*}} = 0. \tag{2.15}$$

Since A_{ω^*} and the orthogonal projections are self-adjoint operators, passing to the adjoint in (2.15) yields

$$\Pi_{\mathcal{Z}_{\omega^*}} A_{\omega^*} \Pi_{\nabla C(X^*)^\perp} = 0. \tag{2.16}$$

Finally, from (2.14)–(2.16) we conclude that

$$S^* A_{\omega^*} S = \begin{pmatrix} \Pi_{\nabla C(X^*)^\perp} A_{\omega^*}|_{\nabla C(X^*)^\perp} & 0 \\ 0 & \Pi_{\mathcal{Z}_{\omega^*}} A_{\omega^*}|_{\mathcal{Z}_{\omega^*}} \end{pmatrix}. \tag{2.17}$$

Next, we note that from (2.10) and (2.11) it follows that

$$\Pi_{\mathcal{Z}_{\omega^*}} A_{\omega^*} h = -\sum_{j,k=1}^{r} (\partial c_k/\partial\omega_j)(\omega^*)\langle h, (\partial X/\partial\omega_k)(\omega^*)\rangle (\partial X/\partial\omega_j)(\omega^*), \tag{2.18}$$

which implies that

$$\sigma\Big(\Pi_{\mathcal{Z}_{\omega^*}} A_{\omega^*}|_{\mathcal{Z}_{\omega^*}}\Big) = \sigma\Big(-(\partial c/\partial\omega)(\omega^*)\Big). \tag{2.19}$$

Since

$$\sigma(\Pi_{\nabla C(X^*)^\perp} A_{\omega^*}|_{\nabla C(X^*)^\perp}) = \sigma(A_{\omega^*}|_{\nabla C(X^*)^\perp}),$$

the theorem follows shortly from Proposition 2.6(i), (2.17) and (2.19). □

2.3. **Variational stability.** Denoting $\ell := \dim\ker A_{\omega^*}$, we strengthen (H2) to

(H2') Near (X^*, ω^*), there is a C^1 family $(X, c)(\omega, s), (\omega, s) \in \mathbb{R}^r \times \mathbb{R}^\ell$,

(2.20) of critical points of (2.2).

With (H3), this implies that, for fixed c, the set of minimizers near X^* forms a C^1 ℓ-dimensional family

$$\{X(\omega(c), s) : s \in \mathbb{R}^\ell\}. \tag{2.21}$$

Differentiating (2.21) with respect to s, noting that $C(X(\omega(c), s)) = c_0$, and recalling (2.4), we find that

$$\mathcal{E}(X(\omega(c), s)) = c_0 \text{ along the family of nearby critical points.} \tag{2.22}$$

Definition 2.8. *A critical point X^* of (2.1) is locally strictly orbitally stable in $\hat{\mathbf{H}} \subset \mathbf{H}$ if, for $X \in \hat{\mathbf{H}}$ in a sufficiently small $\mathbf{H}$-neighborhood of X^*, $\mathcal{E}(X) \geq \mathcal{E}(X^*)$, with equality only if X lies on the set of nearby critical points, and orbitally stable in $\hat{\mathbf{H}} \subset \mathbf{H}$ if, for $X \in \hat{\mathbf{H}}$ in a sufficiently small $\mathbf{H}$-neighborhood of X^*, $\mathcal{E}(X) \geq \mathcal{E}(X^*)$.*

We add the further hypothesis, denoting $\mathbf{H}$-distance from $v \in \mathbf{H}$ to a set $S \subset \mathbf{H}$ by $d_{\mathbf{H}}(v, S)$:

(H4) $\sigma_-(A_{\omega^*}|_{\nabla C(X^*)^\perp}) = 0 \Rightarrow$ for $K > 0$, all $v \in \nabla C^\perp(X^*)$,

$$d_{\mathbf{H}}(v, \ker A_{\omega^*|_{\nabla C^\perp(X^*)}}) \leq K\langle v, A_{\omega^*} v\rangle_{\mathbf{H}}. \tag{2.23}$$

Corollary 2.9. *Assuming* (H1), (H2'), (H3), *and* (H4), *a critical point* X^* *of* (2.1) *is strictly orbitally stable in* $\mathrm{Diff}(\mathcal{E})$ *if and only if*

$$\sigma_-(A_{\omega^*}) = \sigma_-\Big((-\partial c/\partial\omega)(\omega^*)\Big) \tag{2.24}$$

and $\dim\ker A_{\omega_*}|_{\nabla C^\perp(X^*)} = \ell$, *i.e., family* (H2) *is transversal, and orbitally stable in* $\mathrm{Diff}(\mathcal{E})$ *only if*

$$\sigma_-(A_{\omega^*}) = \sigma_-\Big((-\partial c/\partial\omega)(\omega^*)\Big), \tag{2.25}$$

where ω^* *is the (unique) Lagrange multiplier associated with* X^*.

Proof. Immediate, from (2.12), (2.22), and (H4), together with a second-order Taylor expansion like that of Proposition 2.2 along perpendiculars to the manifold (H2) where $C \equiv c_*$. □

Remark 2.10. In the PDE context, Assumption (H4) effectively limits the regularity of the space $\mathbf{H}$. More generally, the $\mathbf{H}$-norm cannot be stronger than any norm under which A_{ω_*} is bounded, or (H4) would fail. If A_{ω_*} is bounded on $\mathbf{H}$ (in particular, if $\mathrm{Diff}(\mathcal{E}) = \mathbf{H}$), then $\mathbf{H}$ is determined. In applications, (H4) typically follows by coercivity of A_{ω_*}, in the form of a Gårding inequality, together with the presence of a spectral gap. For all of the (PDE) examples of this paper, there is a unique space $\mathbf{H}$ on which A_{ω_*} is both bounded and coercive in the strong sense of (H4).

3. Time-evolutionary stability

Corollary 2.9 yields stability results for time-evolutionary systems

$$(d/dt)X = \mathcal{F}(X) \tag{3.1}$$

with variational structure, in particular, for *energy conserving* (e.g., generalized Hamiltonian), and *energy dissipating* (e.g., quasi-gradient type) systems, as we now describe.

For this section, it will be convenient to introduce a graded sequence of Hilbert spaces $\mathbf{H}_1 \subset \mathbf{H} \subset \mathbf{H}_2$, with associated inner products $\langle\cdot,\cdot\rangle_{\mathbf{H}_j}$, $\langle\cdot,\cdot\rangle_{\mathbf{H}}$ of varying strength, and gradients $\nabla_{\mathbf{H}_j}$, satsifying:

(B1)
- (i) For $X \in \mathbf{H}_1$, $\nabla_{\mathbf{H}_2}$ and $\nabla^2_{\mathbf{H}_2}$ of $\mathcal{E}$ and C are well-defined;
- (ii) The eigenfunctions of A_{ω^*} are a basis for $\mathbf{H}$;
- (iii) The negative eigenspace of A_{ω^*} is contained in $\mathbf{H}_1 \subset \mathbf{H}$;
- (iv) $\mathbf{H}_1 \subset \mathrm{Diff}(\mathcal{E})$.

Remark 3.1. Under assumption (B1), if $X^* \in \mathrm{Diff}(\mathcal{E})$ is a critical point of $\mathcal{E}_{\omega^*}$, then

$$\sigma_-(\nabla^2_{\mathbf{H}_2}\mathcal{E}_{\omega^*}(X^*)) = \sigma_-(\nabla^2_{\mathbf{H}}\mathcal{E}_{\omega^*}(X^*)). \tag{3.2}$$

Indeed, we first note that $\sigma_-(\nabla^2_{\mathbf{H}_2}\mathcal{E}_{\omega^*}(X^*)) \le \sigma_-(\nabla^2_{\mathbf{H}}\mathcal{E}_{\omega^*}(X^*))$ trivially holds, since (as bilinear forms) $\nabla^2_{\mathbf{H}}\mathcal{E}_{\omega^*}(X^*)$ is the extension from $\mathbf{H}_1$ to $\mathbf{H}$ of $\nabla^2_{\mathbf{H}_2}\mathcal{E}_{\omega^*}(X^*)$. Under assumption (B1), by computing both associated bilinear forms by eigenfunction expansion, we find that both negative signatures, $\sigma_-(\nabla^2_{\mathbf{H}}\mathcal{E}_{\omega^*}(X^*))$ and $\sigma_-(\nabla^2_{\mathbf{H}_2}\mathcal{E}_{\omega^*}(X^*))$, are equal to the number of negative eigenvalues of A_{ω^*}, considered as an operator on $\mathbf{H}_2$.

3.1. **Generalized Hamiltonian systems with constraints.** We first recall briefly the context considered in [18] of a generalized Hamiltonian system

$$(d/dt)X = J(X)\nabla_{\mathbf{H}_2}\mathcal{E}(X), \quad t \geq 0, \qquad X(0) \in \mathbf{H}, \tag{3.3}$$

$J = -J^*$, which has additional conserved quantities

$$C(X(t)) \equiv c_0 \in \mathbb{R}^r, \text{ i.e., } \langle J(X(t))\nabla C_j(X(t)), \nabla\mathcal{E}(X(t))\rangle \equiv 0, \tag{3.4}$$

$j = 1, \ldots, r$, $t \geq 0$, besides the Hamiltonian $\mathcal{E}(X(t))$, which is always conserved, by skew-symmetric structure

$$\langle \nabla\mathcal{E}(X(t)), J(X(t))\nabla\mathcal{E}(X(t))\rangle = 0. \tag{3.5}$$

Here $J = -J^*$, is assumed to be a closed, densely defined linear operator with X-independent domain.

For such systems, constraints may be divided into two rather different types. The first type, familiar from the classical mechanical case, are constraints $C_j : \mathbf{H} \to \mathbb{R}$ for which

$$g_j(X) := J(X)\nabla C_j(X) \neq 0, \quad \text{for all} \quad j = 1, \ldots, r. \tag{3.6}$$

These are associated with group symmetries $G_j(\theta, \cdot)$ leaving the Hamiltonian $\mathcal{E}$, and thus the flow of (3.3), invariant, generated by the g_j, through

$$(d/d\theta)G_j(\theta; X) = g_j(G_j(\theta; X)), \qquad G_j(0; X) = X, \quad j = 1, \ldots, r. \tag{3.7}$$

When J is one-to-one,[8] these are the only type of conserved quantities that can occur, and every group invariance determines a conserved quantity, following Noether's Theorem, through the relation $\nabla C_j(X) := J^{-1}g_j(X)$, where $g_j(X) := G_j'(0; X)$, $j = 1, \ldots, r$.

When J is not one-to-one, however, as for example in the KdV and Boussinesq cases considered in [18, 60, 9], for which $J = \partial_x$, conserved quantities may or may not correspond to symmetries. For related discussion, see [60]. In this case, we identify a second class of constraints, closer to the themes of this paper, for which

$$J\nabla C_j \equiv 0 \quad \text{for all} \quad j = 1, \ldots, r. \tag{3.8}$$

These are not associated to a nontrivial group symmetry, since $g_j = J\nabla C_j$, $j = 1, \ldots, r$ generates the identity.

For example, in the case considered in [18] of KdV on the whole line, $r = 1$, and the constraint is associated to a group invariance $G(s)$ consisting of spatial translation by distance s. For the generalized KdV case considered in [9] on bounded periodic domains, $r = 2$, only one of the two constraints corresponding to translation invariance and the other corresponding to conservation of mass (integral over one period).

Next, we denote the (possibly trivial) symmetries generated by constraints C_j through (3.7) as $G_j(\theta; X)$, $j = 1, \ldots, r$.

Following [18], we assume that (3.3) has a well-defined solution operator in $\mathbf{H}$ conserving the formal invariants C and $\mathcal{E}$:

(B2) Equation (3.3) admits a (weak) solution $X(t) \in \mathbf{H}$ continuing so long as $\|X(t)\|_{\mathbf{H}} < +\infty$, with $C(X(t))$ and $\mathcal{E}(X(t))$ identically constant.

---

[8] Note: by skew-symmetry, J is one-to-one if it is onto, as sometimes assumed [18]. We remark that J is also Fredholm in the typical situation where we have such non-Noetherian constraints, for example in the case of NLS ($J = \mathrm{i}$) or gKdV ($J = \partial_x$) on bounded periodic domains, in which case J is one-to-one if and only if it is onto.

Definition 3.2. *A generalized traveling-wave solution of* (3.3) *is a solution whose trajectory is contained in the orbit of its initial point* $X(0)$ *under the action of the invariance group generated by* $\{G_j : j = 1, \dots, r\}$.

Lemma 3.3. *A solution emanating from* $X(0) \in \mathrm{Diff}(\mathcal{E})$ *is a generalized traveling wave if and only if* $X(0)$ *is a critical point of the constrained minimization problem* (2.1)*, in which case* $X(0)$ *is a stationary solution of the shifted equation* $(d/dt)X = J(X)\nabla\mathcal{E}_{\omega^*}(X)$*, with associated linearized operator*

$$L_{\omega^*}(X(0)) := d\mathcal{F}_{\omega^*}(X(0)) = J(X(0))A_{\omega^*}. \tag{3.9}$$

Proof. By group invariance of (3.3), it is equivalent that

$$X'(0) = J(X(0))\nabla\mathcal{E}_{\omega^*}(X(0))$$

lie in the tangent space $\mathrm{Span}\{J(X(0))\nabla C_j(X(0)) : j = 1, \dots, r\}$ of the invariance group. Since $\ker J(X(0)) \subset \mathrm{Span}\{\nabla C_j(X(0)) : j = 1, \dots, r\}$, this is equivalent to (2.4), which in turn is equivalent to stationarity as a solution of the shifted equation. Writing, equivalently, $\mathcal{F} = J\nabla\mathcal{E}_{\omega^*}$, differentiating in X, and using (2.4), we obtain (3.9). □

Definition 3.4. *A generalized traveling wave* $X(t) \in \mathbf{H}_1$ *is* $\mathbf{H}_1 \to \mathbf{H}$ *orbitally stable if for every* $\varepsilon > 0$*, there is* $\delta > 0$ *such that, for* $Y(0)$ *within* δ *distance in* $\mathbf{H}_1$ *of* $X(0)$*,* $Y(t)$ *exists for all* $t \geq 0$ *and remains within* ε *distance in* $\mathbf{H}$ *from the orbit of* $X(0)$ *under the group of invariances generated by* $\{G_j : j = 1, \dots, r\}$.

Given a generalized traveling wave solution $X(t)$, we assume further:

(C1) $\quad \ker L_{\omega^*}(X(0)) \subset \mathrm{Span}\{J(X(0))\nabla C_j(X(0)) : j = 1, \dots, r\}$ (transversality),

that is, that all neutral directions are accounted for by group symmetry.

Proposition 3.5. *Assuming* (H1), (H2'),(H3)–(H4), (B1)-(B2), (C1)*, a generalized traveling-wave solution* $X(t)$ *of* (3.3) *is* $\mathbf{H} \to \mathbf{H}$ *orbitally stable if* $X(0) \in \mathbf{H}_1$ *is a strict local orbital minimizer of* (2.1)*, or, equivalently, the signature of* $(\partial c/\partial\omega)(\omega^*)$ *satisfies condition* (2.24)*, and* $\mathrm{Diff}(\mathcal{E}) = \mathbf{H}$.

Proof. By (H1), the level sets of C smoothly foliate $B(X(0), \delta)$, whence it is sufficient to consider perturbations such that $C(Y(0)) = C(X(0))$. As $\mathcal{E}(Y(t)) - \mathcal{E}(X(t))$ is constant, and, by strict orbital minimality, majorizes the $\|\cdot\|_{\mathbf{H}}^2$ distance of $Y(t)$ from the manifold of equilibrium solutions through $X(0)$, which, by (C1), is the group orbit of $X(0)$,[9] we obtain global existence by (B1) and continuation, and ε-boundedness by the assumed continuity of $\mathcal{E}$ with respect to $\|\cdot\|_{\mathbf{H}}$. □

Remark 3.6. The notion of orbital stability defined above does not include phase-asymptotic stability; that is, it does not assert that $Y(t)$ stays close to $X(t)$. It is not difficult to see by specific example (for instance, the different-speed traveling-waves of [18]), that this is all that we can say.

Remark 3.7. Condition (C1) could be weakened to the assumptions that for c near c^* the manifold of critical points (2.21) extends globally as the orbit under the group symmetries generated by $\{G_j : j = 1, \dots, r\}$ of a (for example, compact) manifold of strict local orbital minimizers of (2.1) on which A_ω has a uniform spectral gap, a possibility that does not seem up to now to have been discussed.

Remark 3.8. The "one-way" stability result of Proposition 3.5 recovers the result of [18] for solitary waves of KdV; for J onto (one-to-one), a converse is given in [18] for $r = 1$. In general, time-evolutionary stability does not imply constrained variational stability but only certain parity information on the number of unstable eigenvalues. For complete stability information, a more detailed analysis must be carried out, such as the Krein signature analysis in [9].

[9]Here, we are using group invariance to conclude from local majorization a global property as well; this, precisely, is the step where we require (C1).

3.2. **Quasi-gradient systems with constraints.** We now turn to the situation of our main interest, of *quasi-gradient systems*

$$(d/dt)X = -M(X)\nabla\mathcal{E}(X), \qquad \frac{1}{2}(M+M^*) \geq 0, \tag{3.10}$$

with constants of motion $C(X(t)) \equiv c_0 \in \mathbb{R}^r$ for which

$$(d/dt)\mathcal{E}(X(t)) = \langle \nabla\mathcal{E}(X(t)), M((X(t))\nabla\mathcal{E}(X(t))\rangle \leq 0, \tag{3.11}$$

guaranteed by

$$M\nabla C_j \equiv 0 \quad \text{for all} \quad j = 1, \dots, r, \tag{3.12}$$

similarly as in (3.8). Recall, for M semidefinite,

$$\ker M = \ker M^* \subset \ker \frac{1}{2}(M+M^*),$$

by consideration of the semi-definite quadratic form $\frac{1}{2}(M+M^*)$, so that (3.12) implies also $\langle \nabla C_j, M\nabla\mathcal{E}\rangle \equiv 0$ for all $j = 1, \dots, r$, giving

$$(d/dt)C(X(t)) \equiv 0.$$

For the Keller–Segel (chemotaxis) model considered in [47, 48], $r = 1$, with the single constraint corresponding to conservation of mass (total population). For the equations of viscoelasticity with strain-gradient effects on bounded periodic domains, considered in [44, 58, 6], $r = 2d = 2, 4, 6$, where $d = 1, 2, 3$ is the dimension of allowed deformation directions, with the $2d$ constraints corresponding to conservation of mass in deformation velocity and gradient coordinates.

Lemma 3.9. *$X(t) \equiv X(0) \in \mathrm{Diff}(\mathcal{E})$ is a stationary point of* (3.10) *if and only if it is a critical point of the constrained minimization problem* (2.1), *in which case the associated linearized operator is*

$$L_{\omega^*}(X(0)) := d\mathcal{F}_{\omega^*}(X(0)) = -M(X(0))A_{\omega^*}. \tag{3.13}$$

Proof. Essentially identical to the proof of Lemma 3.3. □

For the applications we have in mind, we find it necessary to substitute for (B2) the milder assumption:

(B2') For data $X(0) \in \mathbf{H}_1$ near an equilibrium solution X^*, (3.1) admits a strong solution $X(t) \in \mathbf{H}_1$, with $\frac{dX}{dt}(t) = \mathcal{F}(X(t)) \in \mathbf{H}_2$. Moreover, this solution continues so long as $X(t)$ remains sufficiently close in $\mathbf{H}$ to an equilibrium.

Remark 3.10. Condition (B2') is satisfied for all of the example quasigradient systems considered in this paper,[10] whereas (B2) is satisfied only for the Cahn–Hilliard equations considered in Section 4.1.

[10] The continuation property follows by nonlinear damping (energy) estimates as introduced in [62, 59], which show that higher Sobolev norms are exponentially slaved to lower ones..

3.2.1. *Orbital stability.* Assume analogously to (C1):

(C1') $\ker L_{\omega^*}$ is generated by group symmetries of (3.10).

Definition 3.11. *A stationary solution $X(t) \equiv X(0) \in \mathbf{H}_1$ of (3.10) is orbitally stable if for every $\varepsilon > 0$, there is $\delta > 0$ such that, for $Y(0)$ within δ distance in $\mathbf{H}_1$ of $X(0)$, $Y(t)$ exists for all $t \geq 0$ and remains within ε distance in $\mathbf{H}$ of the manifold (2.21) of stationary solutions (critical points of (2.1)) near $X(0)$. It is asymptotically orbitally stable if it is orbitally stable and, for $\|Y(0) - X(0)\|_{\mathbf{H}_1}$ sufficiently small, $Y(t)$ converges in $\mathbf{H}$ to the manifold (2.21) of stationary solutions (critical points of (2.1)) near $X(0)$ having the fixed value $c = C(Y(0))$.*

Proposition 3.12. *Assuming* (H1), (H2'), (H3)–(H4), (B1)-(B2'), (C1')*, a stationary solution $X(t) \equiv X(0)$ of (3.10) is $\mathbf{H}_1 \to \mathbf{H}$ orbitally stable if $X(0) \in \mathbf{H}_1$ is a strict local orbital minimizer of (2.1), or, equivalently, the signature of $(\partial c/\partial \omega)(\omega^*)$ satisfies condition* (2.24).

Proof. Identical to that of Proposition 3.5. □

3.2.2. *Asymptotic orbital stability: finite-dimensional case.* Restricting for simplicity to finite dimensions, take without loss of generality $\mathbf{H}_2 = \mathbf{H} = \mathbf{H}_1$, by equivalence of finite-dimensional norms.

Now, make the additional assumption:

(D1) Constant-energy solutions $\mathcal{E}(X(t)) \equiv \mathcal{E}(X(0))$ of (3.10) are stationary.

Lemma 3.13. *A sufficient condition for* (D1) *is $M = M^*$, or, more generally,.*

(D1') $$\ker M = \ker \tfrac{1}{2}(M + M^*),$$

Proof. Using (3.11) we infer

$$\begin{aligned}(d/dt)\mathcal{E}(X(t))|_{t=0} &= -\langle \nabla\mathcal{E}(X(0)), M(X(0))\nabla\mathcal{E}(X(0))\rangle \\ &= -\frac{1}{2}\langle \nabla\mathcal{E}(X(0)), (M + M^*)(X(0))\nabla\mathcal{E}(X(0))\rangle\end{aligned}$$

vanishes if and only if $\nabla\mathcal{E}(X(0))$ lies in

$$\ker\Big(\frac{1}{2}(M + M^*)(X(0))\Big) = \ker M(X(0)),$$

whence

$$\mathcal{F}(X(0)) = -M(X(0))\nabla\mathcal{E}(X(0)) = 0.$$

Noting, finally, that $\mathcal{F}_{\omega^*} \equiv \mathcal{F}$, we are done. □

Remark 3.14. As (D1') shows, (D1) is typical of gradient systems. That is, gradient systems typically do not possess nontrivial constant-energy solutions such as time-periodic solutions or generalized traveling-waves, in contrast with the situation of the Hamiltonian case.

Proposition 3.15. *In finite dimensions, assuming* (H1), (H2'), (H3), (C1')*, and* (D1)*, a stationary solution $X(t) \equiv X(0)$ of (3.10) is $\mathbf{H} \to \mathbf{H}$ asymptotically orbitally stable if $X(0) \in \mathbf{H}$ is a strict local orbital minimizer of (2.1), or, equivalently, the signature of $(\partial c/\partial \omega)(\omega^*)$ satisfies condition (2.24).*

Proof. First note that (H4) and (B1)–(B2') hold always in finite dimensions. By orbital stability, $Y(t)$ remains in a compact neighborhood of the orbit of $X(0)$ under the group symmetries of (3.10), whence, transporting back by group symmetry to a neighborhood of $X(0)$, we obtain an ω-limit set which is invariant up to group symmetry under the flow of (3.10). Moreover, by nonincreasing of $\mathcal{E}(Y(t))$, and local lower-boundedness of $\mathcal{E}$, we find that $\mathcal{E}$ must be constant on the ω-limit set. Combining these properties, we find that points of the ω-limit set must lie on constant-energy solutions, which, by (D1), are stationary points of (3.10). Noting that, up to group symmetry, $Y(t)$ approaches its ω-limit set by definition, we are done. □

Definition 3.16. *A stationary solution $X(t) \equiv X(0) \in \mathbf{H}_1$ of* (3.10) *is $\mathbf{H}_1 \to \mathbf{H}$ stable if for every $\varepsilon > 0$, there is $\delta > 0$ such that, for $Y(0)$ within δ distance in $\mathbf{H}_1$ of $X(0)$, $Y(t)$ exists for all $t \geq 0$ and remains within ε distance in $\mathbf{H}$ of $X(0)$. It is $\mathbf{H}_1 \to \mathbf{H}$ phase-asymptotically orbitally stable if it is stable and, for $\|Y(0) - X(0)\|_{\mathbf{H}_1} \leq \delta$, $Y(t)$ converges time-asymptotically in $\|\cdot\|_{\mathbf{H}}$ to a point $X^\sharp(Y(0))$ on the manifold* (2.21) *of stationary solutions (critical points of* (2.1)*) near $X(0)$.*

Proposition 3.17. *In finite dimensions, assuming* (H1), (H2'),(H3), (D1'), *a stationary solution $X(t) \equiv X(0) \in \mathbf{H}$ of* (3.10) *is $\mathbf{H} \to \mathbf{H}$ stable only if $X(0)$ is a nonstrict local minimizer of* (2.1), *or, equivalently, the signature of $(\partial c/\partial \omega)(\omega^*)$ satisfies condition* (2.25). *It is phase-asymptotically orbitally stable, indeed, time-exponentially convergent, if $X(0)$ is a strict local minimizer of* (2.1), *or, equivalently, the signature of $(\partial c/\partial \omega)(\omega^*)$ satisfies condition* (2.24),

Proof. Again, note that (H4) and (B1)-(B2') hold always in finite dimensions. (i) If $X(0)$ is not a minimizer, then there is a nearby point $Y(0)$ with $C(Y(0)) = C(X(0)) = c^*$ and $\mathcal{E}(Y(0)) < \mathcal{E}(X(0))$. As $C(Y(t)) \equiv C(Y(0))$, the only stationary points to which $Y(t)$ could converge are on the manifold (2.21) with $|s|$ small and $c = c^*$, along which $\mathcal{E} \equiv \mathcal{E}(X(0))$. But, then, by continuity of $\mathcal{E}$ in $\mathbf{H}$, $\mathcal{E}(Y(t))$ would converge to $\mathcal{E}(X(0))$, a contradiction. This establishes the first claim.

(ii) By (H1), the level sets of C smoothly foliate $B(X(0), \delta)$, whence it is sufficient to consider perturbations such that $C(Y(0)) = C(X(0))$. If $X(0)$ is a strict local minimizer, then, in $B(X(0), \varepsilon)$, $\mathcal{E}(Y(t))$ majorizes the $\|\cdot\|$-distance from $Y(t)$ to the manifold of stationary solutions through $X(0)$. Moreover,

$$\begin{aligned}(d/dt)\mathcal{E}(Y(t)) &= \langle \nabla \mathcal{E}(Y(t)), M(Y(t)) \nabla \mathcal{E}(Y(t)) \rangle \\ &\lesssim -\|\Pi_{\nabla C(X(0))^\perp} \nabla \mathcal{E}(Y(t))\|^2 \lesssim -\big(\mathcal{E}(Y(t) - \mathcal{E}(X^\sharp)\big),\end{aligned}$$

where $\Pi_{\nabla C(X(0))^\perp}$ denotes orthogonal projection onto $\nabla C(X(0))^\perp$, and $X^\sharp$ denotes the orthogonal projection of $Y(0)$ onto the set $C(X) = c_0$. Here, the final inequality follows by the assumed variational stability. Thus, $\mathcal{E}(Y(t)) - \mathcal{E}(X(0))$ decays time-exponentially so long as $Y(t)$ remains in $B(X(0), \varepsilon)$, as does therefore $\|\Pi_{\nabla C(X(0))^\perp} \nabla \mathcal{E}(Y(t))\|$. Thus, for some $K, \eta > 0$,

$$\frac{d}{dt} \langle (\partial X/\partial s)|_{s=0,c^*}, Y(t) - Y(0) \rangle \leq K \|M(Y(t)) \nabla \mathcal{E}(Y(t))\| \leq K e^{-\eta t},$$

and so $\langle (\partial X/\partial s)|_{s=0,c^*}, Y(t) - Y(0) \rangle$ converges time-exponentially. which, together with the already established convergence to the manifold of equilibrium solutions, gives stability and phase-asymptotic orbital stability, with convergence at exponential rate. □

3.2.3. *Asymptotic orbital stability: infinite-dimensional case.* Under suitable assumptions guaranteeing compactness of the flow of (3.10), the Lyapunov argument of Proposition 3.15 may be generalized to the infinite-dimensional case, yielding orbital stability with no rate; see [57]. This applies in particular to all of the examples given below, which are (or are equivalent to) systems of parabolic type on bounded domains. However, note that this requires a spectral gap of A_{ω^*}, under which we would expect (assuming a correspondingly nice M, as also holds for our examples), rather, exponential stability, whereas the exponential stability result of Proposition 3.17, first, does not generalize in simple fashion to infinite dimensions, due to the presence of multiple norms, and, second, requires the restrictive assumption (D1'), which does not apply the important example of viscoelasticity, below.

Moreover, there are interesting situations in which the essential spectrum of A_{ω^*} and $d\mathcal{F}$, extend to the imaginary axis; for example, stability of solitary waves in viscoelastic, Cahn–Hilliard, or chemotaxis systems, as studied respectively in [61, 29], [26, 28], and [48]. In such situations, it has proven useful to separate the questions of spectral and nonlinear stability; see [59] for a general discussion of this strategy. In particular, for each of the above-mentioned settings, it can be shown that to show nonlinear orbital stability, with time-algebraic rates of decay, it is enough to establish transversality plus strict stability of point spectrum, whereupon essential spectrum and nonlinear stability may be shown by separate analysis.[11]

For both of these reasons, we restrict ourselves in the infinite-dimensional setting to the simpler treatment of stability of point spectrum of $d\mathcal{F}$, from which the relevant nonlinear stability result may in most settings (in particular, all of those mentioned in this paper) may be readily deduced.

Definition 3.18. *A stationary solution $X(t) \equiv X(0) \in \mathbf{H}_1$ of* (3.10) *has orbitally stable point spectrum if (i) the dimension of the zero eigenspace of $d\mathcal{F}(X(0)) = M(X(0))A_{\omega^*}$ is equal to the dimension* $\dim\ker A_{\omega^*} + \dim c$ *of the manifold* (2.21) *of stationary solutions (transversality), and (ii) the nonzero point spectrum of $d\mathcal{F}(X(0))$ has strictly negative real part.*

Definition 3.19. *A stationary solution $X(t) \equiv X(0) \in \mathbf{H}_1$ of* (3.10) *is $\mathbf{H}_1 \to \mathbf{H}$-linearly asymptotically orbitally stable if, for $Y(0) \in H_1$, the solution $Y(t)$ of $(d/dt)Y = M(X(0))A_{\omega^*}Y$ converges in $\mathbf{H}$ to the tangent manifold of the manifold* (2.21) *of stationary solutions (critical points of* (2.1)*).*

Define now the following local linear analog of the nonlinear condition (D1):

(E1) The center subspace of $M(X(0))A_{\omega^*}$ consists of $\ker M(X(0))A_{\omega^*}$;

that is, 0 is the only pure imaginary eigenvalue of $M(X(0))A_{\omega^*}$, and associated eigenvectors are genuine.

Lemma 3.20. *Sufficient conditions for* (E1) *are* (D1') *(e.g., $M = M^*$) and $A_{\omega^*}|_{\nabla C(X(0))^\perp} \geq 0$.* (D1') *by itself implies that there are no nonzero imaginary eigenvalues of $M(X(0))A_{\omega_*}$.*

Proof. Let $M(X(0))A_{\omega^*}Y = i\mu Y$, $\mu \in \mathbb{R}$. Then, dropping ω^* and $X(0)$, $\langle AY, MAY\rangle = i\mu\langle Y, AY\rangle$, hence

$$\langle AY, (M + M^*)AY\rangle = 2\mathrm{Re}\langle AY, MAY\rangle = 0,$$

and, by semidefiniteness of $(M + M^*)$, we conclude that $(M + M^*)AY = 0$. By (D1'), this implies $MAY = 0$. This shows that 0 is the only pure imaginary eigenvalue of MA.

Now, suppose that $(MA)^2Y = 0$. Then, by (D1'), $M^*AMAY = 0$, hence $\langle MAY, A(MAY)\rangle = 0$. As $MAY \in \nabla C(X(0))^\perp$, by (3.12), and $A|_{\nabla C(X(0))^\perp} \geq 0$ by assumption, this implies that $MAY \in \ker A$, hence $\langle AY, (M + M^*)AY\rangle = 0$ and so $AY \in \ker(M + M^*) = \ker M$ and $MAY = 0$ as before. This shows that there are no generalized eigenvectors in the zero eigenspace of MA, completing the proof. □

Remark 3.21. Transversality condition (i) of Definition 3.18 concerns well-behavedness of the existence problem. Under (E1), it is guaranteed by our hypotheses (H1)–(H4), through (2.10), and the associated relation

$$\ker M(X(0))A_{\omega_*} = \ker A_{\omega_*} \oplus \mathrm{Span}\{\partial X/\partial\omega_j)(\omega_*) : j = 1, \ldots, r\}.$$

Along with (E1), we identify the following mild properties of the linearized solution theory:[12]

(E2) If $X(0)$ is asymptotically orbitally stable, then the solutions of equation $\dfrac{dY}{dt} = M(X(0))A_{\omega^*}Y$ with $Y(0)$ in the constrained unstable subspace of A_{ω_*} exist for all time and converge in $\mathbf{H}$ to the center

[11] In the periodic parabolic examples studied below, nonlinear stability follows easily from spectral stability by general analytic semigroup theory [21, 51].

[12] This is satisfied for all of the examples of this paper, not only on the unstable subspace, but all of $\mathbf{H}$ as expected.

subspace of $M(X(0))A_{\omega^*}$.

(E3) $\qquad$ A_{ω_*} is bounded on $\mathbf{H}$ as a quadratic form.

Theorem 3.22. *Assuming* (H1), *(*H2'),(H3)–(H4), (B1)-(B2'), *and* (E1), *a stationary solution* $X(t) \equiv X(0) \in \mathbf{H}$ *of* (3.10) *has orbitally stable point spectrum if and is* $\mathbf{H} \to \mathbf{H}$ *linearly orbitally asymptotically stable only if* $X(0)$ *is a strict local minimizer of* (2.1)*, or, equivalently, the signature of* $(\partial c/\partial \omega)(\omega^*)$ *satisfies condition* (2.24)*. Assuming also* (E2)–(E3)*,* $X(0)$ *has strictly orbitally stable point spectrum if and only if* $X(0)$ *is strictly orbitally stable, or* (2.24)*.*

Proof. ($\Leftarrow$) (i) Dropping ω^* and $X(0)$, $MAY = 0$ is equivalent to $AY = \sum_{j=1}^r \alpha_j \nabla C_j(X(0))$, for some $\alpha \in \mathbb{C}^r$. Since $A\frac{\partial X}{\partial \omega_j}(\omega^*) = -\nabla C_j(X(0))$ for all $j = 1, \ldots, r$, we thus have

$$A\Big(Y + \sum_{j=1}^r \alpha_j \frac{\partial X}{\partial \omega_j}(\omega^*)\Big) = 0,$$

or $Y \in \mathrm{Span}\{\partial X/\partial \omega_j(\omega^*) : j = 1, \ldots, r\} \oplus \mathrm{Span}\{\nabla C_j(X(0)) : j = 1, \ldots, r\}$ as claimed.

(ii) By (E1), it is sufficient to show that $\mathrm{Re}\lambda \leq 0$ whenever we assume that $-M(X(0))A_{\omega^*}Y = \lambda Y$. Dropping ω^* and $X(0))$ again, we find from $-\mathrm{Re}\langle AY, MAY\rangle = \mathrm{Re}\lambda \, \langle Y, AY\rangle$ that either $AY = 0$, hence $\lambda = 0$, or $\mathrm{Re}\lambda = \frac{-\langle AY, MAY\rangle}{\langle Y, AY\rangle} \leq 0$.

($\Rightarrow$) On the other hand, suppose $X(0)$ is not strictly orbitally stable. Then, either $A_{\omega^*}|_{\nabla C(X(0))^\perp}$ has a zero eigenvector not in the tangent subspace $\frac{\partial X}{\partial s}$ of the family of equilibria described in (2.21), violating transversality, or else $\check{\mathcal{E}}(Y(0)) := \langle Y(0), A_{\omega^*}Y(0)\rangle < 0$ for some $Y(0) \in \nabla C(X(0))^\perp$, in which case we see from $(d/dt)\langle Y(t), A_{\omega^*}Y(t)\rangle \leq 0$ that $\check{\mathcal{E}}(Y(t)) < 0$ for all $t \geq 0$ for the solution Y of the linearized equations with initial data $Y(0)$. Thus, $Y(t)$ cannot approach the zero-$\check{\mathcal{E}}$ set tangent to the manifold of nearby minimizers as $t \to \infty$, violating linear asymptotic orbital stability. Assuming also (E2), so that we have transversality by Remark 3.21, we find that there are strictly unstable spectra of $M(X(0))A_{\omega_*}$, or else (E1)–(E3) would yield convergence to the manifold of nearby minimizers, a contradiction. □

Remark 3.23. It would be interesting to establish a relation between the number of constrained unstable eigenvalues of A_{ω^*} and the number of unstable eigenvalues $n_-(M(X(0))A_{\omega^*})$ of the operator $-M(X(0))A_{\omega^*}$, similar to those obtained for A_{ω^*} and $J(X(0))A_{\omega^*}$ in the Hamiltonian case (see, e.g., [18, 9]). These are equal (under appropriate assumptions guaranteeing that eigenvectors remain in appropriate spaces under the action of $M(X(0))^{\pm 1/2}$) when $M(X(0))$ is symmetric positive definite, by the similarity transformation

$$M(X(0))A_{\omega^*} \to M(X(0))^{-1/2}M(X(0))A_{\omega^*}M(X(0))^{1/2} = M(X(0))^{1/2}A_{\omega^*}M(X(0))^{1/2}.$$

More generally, they are equal if $M(X(0)) + M(X(0))^* > 0$, by homotopy to the symmetric case, together with the observation that, by (D1') and Lemma 3.20, zero is the only possible imaginary eigenvalue of $M(X(0))A_{\omega^*}$, and, since $M(X(0))A_{\omega^*}M(X(0))A_{\omega^*}Y = 0$ implies that $A_{\omega^*}M(X(0))A_{\omega^*}Y = 0$ and thus $\mathrm{Re}\langle A_{\omega^*}Y, M(X(0))A_{\omega^*}Y\rangle = 0$, hence $Y \in \ker A_{\omega^*}$, from which observations we may deduce that no eigenvalues cross the imaginary axis throughout the homotopy.

We conjecture that, more generally, under assumptions (H1), (H2'), (H3) and (D1') the following assertion holds true:

$$n_-(M(X(0))A_{\omega^*}) = \sigma_-(A_{\omega^*}|_{\nabla C^\perp(X(0))}) = \sigma_-(A_{\omega^*}) - \sigma_-(dc/d\omega)(\omega_*)). \tag{3.14}$$

Here we note that (H3) is needed only for the second equality. In the finite-dimensional, diagonalizable, case, this follows easily from the observation that $M(X(0))A_{\omega^*}Y = \mu Y$, $\mathrm{Re}\mu \gtrless 0$ implies that $Y \in \nabla C(X(0))^\perp$, $Y \notin \ker A_{\omega^*}$, and $\langle A_{\omega^*}Y, (M(X(0)) + M(X(0))^*)A_{\omega^*}Y\rangle > 0$, and, moreover, $\mathrm{Re}\langle A_{\omega^*}Y, M(X(0))A_{\omega^*}Y\rangle = \mathrm{Re}\mu\langle Y, A_{\omega^*}Y\rangle$, so that $\langle Y, A_{\omega^*}Y\rangle \gtrless 0$, whence

$$n_\pm(M(X(0))A_{\omega^*}) \le \sigma_\pm\Big(A_{\omega^*}|_{\nabla C(X(0))^\perp}\Big).$$

Recalling that (H1),(H2') (via (2.9)) imply that

$$n_0(M(X(0))A_{\omega^*}) = \dim\ker M(X(0))A_{\omega^*} = \sigma_0(A_{\omega^*}) + \dim\ker M(X(0)),$$

we thus obtain the result. Galerkin approximation [13] then yields the result in the case that $M(X(0))A_{\omega^*}$ has a spectral gap and finitely many stable eigenvalues.

4. Examples: applications to co-periodic stability of periodic waves

4.1. **Cahn–Hilliard systems.** First, we consider the Cahn–Hilliard system

$$u_t = \partial_x^2(\nabla W(u) - \partial_x^2 u) \tag{4.1}$$

where $u \in \mathbb{R}^m$ and $W : \mathbb{R}^m \to \mathbb{R}$ is a smooth function. This model can be expressed in the form (3.10) with

$$\mathcal{E}(u) := \int_0^1 \Big(\frac{1}{2}|u_x|^2 + W(u)\Big)dx \quad \text{and} \quad M := -\partial_x^2 \ge 0 \tag{4.2}$$

$$\nabla_{L^2}\mathcal{E}(u) = \nabla W(u) - \partial_x^2 u. \tag{4.3}$$

Note that scaling x, t, and W, we can restrict to the unit interval without loss of generalization. We note that the energy is continuously differentiable on $H^1_{\mathrm{per}}[0,1]$ and L^2-gradients are continuous on $H^2_{\mathrm{per}}[0,1]$. The system fits our framework so that we can derive stability and instability results that recover and substantially extend the spectral stability conclusions of [26] for spatially periodic solutions of (4.1). We define the constraint function by

$$C(u) = \int_0^1 u(x)dx.$$

To find solutions of (2.4) for the Cahn-Hilliard case, we need to solve the nonlinear oscillator problem

$$-u_{xx} + \nabla W(u) = -\omega^*, \tag{4.4}$$

with $\omega^* \in \mathbb{R}^m$. Whenever the potential W is not convex, (4.4) possesses families of periodic solutions which are in fact critical points of (4.1). For the linearization at such periodic solutions, we find simply the second derivative of the energy since the constraint function C is linear:

$$A_{\omega^*} = \nabla^2_{L^2}\mathcal{E}(u^*) - \omega^*\nabla^2_{L^2}C(u^*) = -\partial_x^2 + \nabla^2 W(u^*). \tag{4.5}$$

In the scalar case, $m = 1$, consider periodic patterns with minimal period 1, first. These possess precisely 2 sign changes, so that by Sturm-Liouville theory, the translational eigenfunction u^*_x is either the second or third eigenvalue of A_{ω^*}. More precisely, one can decompose the spatially periodic linearized problem on $[0,1]$ into a problem on even and odd functions, which, equivalently, satisfy the linearized problem on $[0,1/2]$ with Neumann and Dirichlet conditions, respectively. Since u^*_x has a sign on $[0,1/2]$, the linearization is semi-definite on the Dirichlet subspace. The Neumann problem can have $\sigma_- = 1$ or $\sigma_- = 2$. It is well-known that the index in the Neumann case depends on the period-amplitude relation. In fact, periodic orbits patterns in the scalar case come in families parameterized by the maximum $u_{\max}$ of the solution, once we allow the spatial period L to vary. Whenever the period is increasing with amplitude, we find $\sigma = -1$, decreasing period corresponds to $\sigma = -2$. We refer the reader to [35, 52] and references therein for proofs

[13] with well-chosen subspaces including $\ker(M(X(0))A_{\omega^*})$

in the analogous Dirichlet case. Alternatively, this could be (but to our knowledge has not been) deduced by a co-periodic stability index computation like those in [44, 26] in the conservation law case, which would yield that the Morse index be even or odd accordingly as $du_{\max}/dL$ (related by a nonvanishing factor to the derivative of the Evans function at $\lambda = 0$) is less than or greater than zero. In particular, we find that periodic patterns are co-periodically stable if and only if $du_{\max}/dL > 0$ *and* $dc/d\omega > 0$.

In the system case, we obtain a partial analog, namely, a necessary and sufficient condition for co-periodic stability of periodic solutions of (4.1) is that the number of positive eigenvalues of $(\partial c/\partial\omega)(\omega^*)$ is equal to the number of negative eigenvalues of $A_{\omega^*} = -\partial_x^2 + \nabla^2 W(u^*)$, considered as an operator on all of $L^2[0,1]_{per}$ (i.e., with no zero-mass restriction). *This result appears to be new,* despite the considerable attention to Cahn–Hilliard equations in the pattern-formation and phase-transition literature.

Moreover, for the important case of multiply-periodic solutions in $\mathbb{R}^m$, which appear (see [27]) not to have been treated so far either in the scalar or system case, we obtain corresponding stability conditions by exactly the same argument. Specifically, we find that a necessary and sufficient condition for co-periodic stability is that the number of positive eigenvalues of $(\partial c/\partial\omega)(\omega^*)$ is equal to the number of negative eigenvalues of eigenvalues as does $A_{\omega^*} = -\partial_x^2 + \nabla^2 W(u^*)$, considered as an operator on $L^2[0,1]_{per}$. *These results too appear to be new.*

Finally, we collect the results of this subsection in the following proposition.

Proposition 4.1. *Assume that u^* is a* ***periodic or multi–periodic*** *pattern of* (4.1). *The following assertions hold true:*

(i) *If $m = 1$ and $x \in [0,1]_{\text{per}}$, the pattern u^* is stable if and only if $(dc/d\omega)(\omega^*) > 0$ and $dL/du_{\max} > 0$;*
(ii) *If $m > 1$ or for multi-periodic patterns, u^* is stable if and only if $-(\partial c/\partial\omega)(\omega^*)$ has the same number of negative eigenvalues as does $A_{\omega^*} = -\partial_x^2 + \nabla^2 W(u^*)$, considered as an operator on $L^2[0,1]_{per}$.*

Remark 4.2. Integrating the traveling-wave ODE (4.4) over one period, we find further that

$$\omega^* = -\nabla W(u^*)^a, \tag{4.6}$$

where superscript a denotes average over one period of the periodic solution u^*, here indexed by its mean $c = \int_0^1 u^*(x)dx$. This gives the more explicit stability condition

$$d(W'(u^*)^a)/dc < 0 \text{ (case } m = 1) \tag{4.7}$$

or $\sigma_-\big(\partial(\nabla W(u^*)^a)/\partial c\big) = \sigma_-\big(A_{\omega^*}\big)$ (case $m > 1$).

Remark 4.3. By (4.6), $(dc/d\omega)(\omega^*)$ simplifies in the long-wavelength limit toward a homoclinic solution with endstate u^∞ as $x \to \pm\infty$ simply to $-W''(u^\infty)$, which for $m = 1$ (by the existence criterion that u^∞ be a saddle point) is scalar and negative, implying co-periodic instability. Stable, large-wavelength patterns can be found in the heteroclinic limit, when the periodic solutions limit on a heteroclinic loop. In the small-amplitude limit, that is, for almost constant periodic solutions, both coperiodic stability and instability can occur. In the simple scalar cubic case, $W'(u) = L^2(-u + u^3)$, small-amplitude patterns exist for $|c| < 1/\sqrt{3}$ for an appropriate length parameter L. These small amplitude patterns are stable when $|c| < 1/\sqrt{5}$ and unstable when $|c| > 1/\sqrt{5}$; see [20].

4.1.1. *Sturm-Liouville for systems.* It is a natural question to ask whether one could obtain information about $\sigma_-(A_{\omega^*})$ also in the system case $m > 1$ using generalizations of Sturm-Liouville theory such as the Maslov index. As described in [2, 15, 55], there exist such generalizations not only for operators of the form A_{ω^*}, but for essentially arbitrary ordinary differential operators A_{ω^*}

of self-adjoint type. However, so far as we know, all of these generalizations involve information afforded by evolving planes spanned by d different solutions, rather than the single zero eigenfunction u_x^* from which we wish to draw conclusions. Indeed, in the absence of additional special structure (as described for example in Cor. 5.2), we do not know how to deduce such information analytically.

On the other hand, we note that this point of view does suggest an intriguing numerical approach to this problem. Namely, borrowing the point of view espoused in [15, 55] that Sturm theory amounts to the twin properties of monotonicity in Morse index of the operator $A_{\omega^*} - \lambda \mathrm{Id}$ on domain $[0, x]$ with respect to frequency λ and spatial variable x, we find that we may compute the number of eigenvalues $\lambda < \lambda_0$ of A_{ω^*} on domain $[0, x_0]$ by counting instead the number of *conjugate points* $x \in (0, x_0)$ of $A_{\omega^*} - \lambda_0 \mathrm{Id}$, i.e., values for which $A_{\omega^*} - \lambda_0 \mathrm{Id}$ has a kernel on $[0, x]$. This in turn may be computed using the same *periodic Evans function* of Gardner [16], as described for example in [4]. that would be used to study behavior in λ. Some apparent advantages of this approach are that (i) it avoids the computationally difficult large-$|\lambda|$ regime, and (ii) the computational expense is no greater in computing values of the Evans function along all of $[0, x_0]$ than in evaluating at the single value $x = x_0$, which in any case requires integrating the eigenvalue ODE over the whole interval. This seems an interesting direction for further investigation.

4.2. **Coupled conservative–reaction diffusion equations.** We consider the general model

$$\begin{aligned} u_t &= \partial_x \Big(a(u) u_x - b(u) v_x \Big), \\ v_t &= v_{xx} + \delta u + g(v), \end{aligned} \tag{4.8}$$

where a, b, g are of class C^3 and $a(u) \geq a_0 > 0$, $b(u) \geq b_0 > 0$, $u, v \in \mathbb{R}$, $\delta > 0$. We note that this system can be expressed in the form (3.10) with

$$\mathcal{E}(u, v) := \int_0^1 \Big(F(u) - uv - \delta^{-1} G(v) + (2\delta)^{-1} |\partial_x v|^2 \Big) dx, \tag{4.9}$$

where $F''(u) := \frac{a(u)}{b(u)}$ and $G'(v) = g(v)$, and

$$\begin{aligned} M(u, v) &:= - \begin{pmatrix} \partial_x \left(b(u) \partial_x \right) & 0 \\ 0 & \delta \end{pmatrix} \geq 0, \\ \nabla_{L^2} \mathcal{E}(u, v) &= \begin{pmatrix} F'(u) - v \\ -\delta^{-1} \partial_x^2 v - u - \delta^{-1} g(v) \end{pmatrix}. \end{aligned} \tag{4.10}$$

This general form of a conservation law coupled to a scalar reaction-diffusion equation includes a variety of interesting models such as the Keller-Segel equations of chemotaxis and many of their variations, reaction-diffusion systems with stoichiometric conservation laws, super-saturation models for recurrent precipitation and Liesegang patterns, as well as phase-field models for phase separation and super-cooled liquids; see for instance [33, 34, 48, 49].

Again, these systems fit into our general framework of quasi-gradient systems and we can relate spectral stability to signatures of symmetric forms. Since the system (4.8) is parabolic, spectral stability will readily imply nonlinear asymptotic stability up to spatial translations. We thereby recover and extend stability computations from [47].

In this case the constraint function is given by

$$C(u, v) = \int_0^1 u(x) dx.$$

Next, we look for solutions of system (2.4). We note that in the case at hand this system is

$$\begin{aligned} &F'(u) - v = -\omega^*, \\ &-\tfrac{1}{\delta}\Big(v_{xx} + g(v)\Big) - u = 0. \end{aligned} \tag{4.11}$$

Since $F'' > 0$, we can solve the first equation of (4.11) for u. We then substitute the solution $u = \varphi(v - \omega^*)$ in the second equation of (4.11), and conclude that the v–component of any solution of (4.11) satisfies the nonlinear Schrödinger equation

$$v_{xx} + \delta\varphi(v - \omega^*) + g(v) = 0. \tag{4.12}$$

The above ω^*-family of equations is a parametrized nonlinear oscillator, well-known to support, in appropriate circumstances, a variety of homoclinic, heteroclinic, and periodic orbits.

As the constraint function C is linear, we find that

$$A_{\omega^*} = \nabla^2\mathcal{E}(u^*, v^*) - \omega^*\nabla^2 C(u^*, v^*) = \begin{pmatrix} \frac{a(u^*)}{b(u^*)} & -1 \\ -1 & -\frac{1}{\delta}\Big(\partial_x^2 + g'(v^*)\Big) \end{pmatrix}. \tag{4.13}$$

In analogy to the case of the Cahn-Hilliard equation, we can give necessary and sufficient condition for co-periodic stability of spatially periodic patterns. Denote by $v_{\max}$ the maximum of v and write $L(v_{\max})$ for the period of the family of periodic solutions to (4.12).

Proposition 4.4. *The necessary and sufficient condition for co-periodic stability of spatially periodic patterns of coupled conservative–reaction diffusion equations* (4.8) *is* $(dc/d\omega)(\omega^*) > 0$ and $dL/dv_{\max} > 0$.

Proof. To start, we consider (u^*, v^*) a co-periodic solution of (4.11) of period 1. We note that all we need to show is that $\sigma_-(A_{\omega^*}) = 1$. Since $\varphi = (F')^{-1}$ and $u^* = \varphi(v^* - \omega^*)$, we obtain that

$$\varphi'(v^* - \omega^*) = \frac{1}{F''(\varphi(v^* - \omega^*))} = \frac{1}{F''(u^*)} = \frac{b(u^*)}{a(u^*)}. \tag{4.14}$$

Since v^* is the solution of the nonlinear oscillator (4.12) we can again conclude that $\tilde{A}_{\omega^*} := -\frac{1}{\delta}\Big(\partial_x^2 + g'(v^*)\Big)$,

$$\tilde{A}_{\omega^*} - \frac{b(u^*)}{a(u^*)} \quad \text{has only one simple, negative eigenvalue,} \tag{4.15}$$

provided that $dL/dv_{\max} > 0$.

One can readily check that A_{ω^*} has the following decomposition:

$$A_{\omega^*} = S^* T_{\omega^*} S, \quad \text{where} \tag{4.16}$$

$$S = \begin{pmatrix} \frac{a(u^*)}{b(u^*)} & 0 \\ -1 & 1 \end{pmatrix} \quad \text{and} \quad T_{\omega^*} = \begin{pmatrix} \frac{b(u^*)}{a(u^*)} & 0 \\ 0 & \tilde{A}_{\omega^*} - \frac{b(u^*)}{a(u^*)} \end{pmatrix}. \tag{4.17}$$

Since S_{ω^*} is bounded on $L^2(\mathbb{R}, \mathbb{C}^2)$, invertible with bounded inverse, we infer from Proposition 2.5 that $\sigma_-(A_{\omega^*}) = \sigma_-(T_{\omega^*})$. From (4.15) and since the functions a and b are positive, we infer that there is $\overline{\psi} \in L^2(\mathbb{R}) \times H^2(\mathbb{R})$, $\overline{\psi} \neq 0$, such that $T_{\omega^*}\overline{\psi} = \lambda\overline{\psi}$ for some $\lambda < 0$ and the restriction of $T_{\omega^*}|_{\{\overline{\psi}\}^\perp} \geq 0$. From Proposition 2.6(i) conclude that $\sigma_-(A_{\omega^*}) = \sigma_-(T_{\omega^*}) = 1$. □

Remark 4.5. *Similarly as in Remark 4.2, we obtain, integrating traveling-wave equation* (4.11),

$$\omega^* = -(F'(u) - v)^a, \tag{4.18}$$

where superscript a denotes average over one period, along with relation

$$g(v)^a = -\delta u^a = -\delta c. \tag{4.19}$$

This gives the explicit co-periodic stability condition $(d/dc)(F'(u^*) - v^*)^a < 0$.

Again, one can simplify in the homoclinic, long-wavelength limit with endstate $(u,v)^\infty$ *as* $x \to \pm\infty$ *to*

$$(d/du^\infty)(F'(u^\infty) - v^\infty) < 0, \text{ where } v^\infty = -\delta g^{-1}(u^\infty). \tag{4.20}$$

Comparing with the reduced traveling-wave ODE (4.12)*, we see (cf.* [48]*) that this is the requirement that* (u^∞, v^∞) *be a saddle (center), hence by the associated existence theory we obtain again co-periodic instability in the homoclinic limit. In the limit towards constant states, bifurcations can be more complicated and both co-periodic stability and instability can occur* [19]*.*

Remark 4.6. Wolansky [56] gives a natural extension of (4.8) to chemotaxis systems that are "not in conflict" in a certain sense, with $u \in \mathbb{R}^m$, $v \in \mathbb{R}^n$, m and n arbitrary. For this class of systems, all of our results apply, giving a generalization of the stability condition to the system case.

Remark 4.7. We mention also a more general class of chemotaxis systems identified by Horstman [25], possessing a decreasing Lyapunov function $\mathcal{E}$, but apparently not coercive in the sense of (D1'). It would be interesting to see whether this class of systems could be treated similarly as the case of viscoelasticity with strain-gradient effects, below.

4.3. **Viscoelasticity with strain-gradient effects.** The equations of planar viscoelasticity with strain-gradient effects, written in Lagrangian coordinates, appear [5, 58, 6] as

$$\begin{aligned} \tau_t - u_x &= 0, \\ u_t - (\nabla W(\tau))_x &= (b(\tau)u_x)_x - (h(\tau_x)\tau_{xx})_x. \end{aligned} \tag{4.21}$$

Here $\tau, u \in \mathbb{R}^d$, $d = 1,2,3$, $b : \mathbb{R}^d \to \mathbb{R}$ is a smooth function, $b(\tau) \geq b_0 > 0$ and $h(\sigma) = \nabla^2\Psi(\sigma) \geq h_0 I_d > 0$, for some smooth function $\Phi : \mathbb{R}^d \to \mathbb{R}$. Standard choices of b and h are [5, 58]

$$b(\tau) = \tau_3^{-1}\text{Id}, \quad h(\tau_x) = \text{Id}, \tag{4.22}$$

b corresponding to the usual Navier–Stokes stress tensor, and h to the phenomenological energy $\psi(p) = |p|^2/2$ of the usual Cahn–Hilliard theory of phase transitions.

Again, we consider this system as a quasi-gradient system with constraints within the framework (3.10), where the energy is given through

$$\mathcal{E}(\tau, u) := \int_0^1 \left(\frac{|u|^2}{2} + W(\tau) + \Psi(\tau_x) \right)(x)dx \tag{4.23}$$

and

$$M(\tau, u) := -\begin{pmatrix} 0 & \partial_x \\ \partial_x & \partial_x(b(\tau)\partial_x) \end{pmatrix} \geq 0. \quad \nabla_{L^2}\mathcal{E}(\tau, u) = \begin{pmatrix} \nabla W(\tau) - h(\tau_x)\tau_{xx} \\ u \end{pmatrix}. \tag{4.24}$$

The constraints of the system are given by

$$C(\tau, u) = \Big(\int_0^1 \tau(x)dx, \int_0^1 u(x)dx \Big).$$

Further, if (τ^*, u^*) is a solution of (2.4), using that the constraint function C is linear, we compute

$$A^*_\omega = \begin{pmatrix} -\nabla^2\Psi(\tau^*_x)\partial_x^2 + \nabla^2 W(\tau^*) & 0 \\ 0 & 1 \end{pmatrix}. \tag{4.25}$$

Looking for solutions of system (2.4), in this case we note that the u-component should be constant, that is $u = -\omega_u \in \mathbb{R}^d$. The τ-component satisfies the equation

$$\nabla W(\tau) - \big(\nabla\Psi(\tau_x)\big)_x + \omega_\tau = 0, \tag{4.26}$$

with $\omega_\tau \in \mathbb{R}^d$. The above equation is a nonlinear oscillator, which under appropriate conditions on W and Ψ (for example, Ψ close to the identity, W not convex, or, more generally, when the matrix pencil $\nabla^2\Psi(\tau_*)\rho - \nabla^2 W(\tau_*)$ possesses a negative eigenvalue ρ) is also known to have a variety of solutions depending smoothly on ω_τ, proving that (4.21) has a $\omega = (\omega_\tau, \omega_u) \in \mathbb{R}^{2d}$ dependent family of critical points, constant in the u-component.

Note ([61, 5, 58]) that

$$\frac{d}{dt}\mathcal{E}(\tau, u) = -\int_0^1 \langle u_x(x), b(\tau(x))u_x(x)\rangle\, dx \tag{4.27}$$

is not coercive, according to the fact that

$$\frac{1}{2}(M + M^*)(\tau, u) = -\begin{pmatrix} 0 & 0 \\ 0 & \partial_x(b(\tau)\partial_x) \end{pmatrix}$$

has a larger (indeed, infinite-dimensional) kernel than that of M. However, it is an interesting fact that if $\frac{d}{dt}\mathcal{E}(\tau, u) = 0$, then $u_x \equiv 0$ by (4.27), whence $\partial_x^r \tau_t \equiv 0$, all $r \geq 0$, by (4.21)(i), and so $\frac{d}{dt}^2\mathcal{E}(\tau, u) = \|u_t\|^2$. Thus, $\frac{d}{dt}\mathcal{E} \equiv 0$ gives $\frac{d}{dt}(\tau, u) \equiv 0$, so that constant-energy solutions are stationary as in the coercive case (D1'). Likewise, we have the corresponding linear property:

Lemma 4.8. *The condition* (E1) *is satisfied at any variationally stable critical point of* (4.21).

Proof. Denoting $Y = \begin{pmatrix} \tau \\ u \end{pmatrix}$, assuming $MA_{\omega^*}Y = i\mu Y$ and dropping ω^*, we find that

$$MAY = i\mu Y \Rightarrow \mathrm{Re}\langle AY, \frac{1}{2}(M + M^*)AY\rangle = 0 \Rightarrow \partial_x u = 0,$$

whence $i\mu\tau = u_x = 0$, so that $\mu = 0$ or else $\tau \equiv 0$ and $u \equiv$ constant, giving $MAY = 0$ and again $\mu = 0$. Thus, zero is the only pure imaginary eigenvalue of MA_{ω^*}. Now, suppose that Y is a generalized zero-eigenvector, $(MA)^2Y = 0$, or $AMAY \in \ker M$, i.e., $\partial_x AMAY = 0$. Computing

$$\partial_x AMA = \partial_x \begin{pmatrix} 0 & -(\partial_x^2 - \nabla^2 W(\tau^*))\partial_x \\ -\partial_x(\partial_x^2 - \nabla^2 W(\tau^*)) & \partial_x(b(\tau^*)\partial_x) \end{pmatrix}, \tag{4.28}$$

$\partial_x AMAY = 0$ yields, in the τ-coordinate, $\partial_x(\partial_x^2 - \nabla^2 W(\tau^*)\partial_x u = 0$, hence, by the assumed definiteness of $\partial_x^2 - \nabla W(\tau^*)$ on $\ker \Pi_{\nabla C(X^*)^\perp}$, that $\partial_x u = 0$. Substituting this into the u-coordinate of (4.28) then yields $-\partial_x^2(\partial_x^2 - \nabla W(\tau^*))\tau = 0$, hence $-\partial_x(\partial_x^2 - \nabla W(\tau^*))\tau = 0$. Combining this with $\partial_x u = 0$, we obtain $MAY = 0$, and so Y is a genuine eigenfunction. This verifies that all zero-eigenfunctions are genuine, completing the proof. □

As a corollary, we obtain spectral stability whenever (2.24) is satisfied, from which stability follows by the general theory developed in [58, 6, 29, 36, 59] and references therein. Notably, this holds both for periodic solutions, on $H^1[0,1]_{\mathrm{per}}$ or for front and pulse type solutions on $H^1(\mathbb{R})$. We note that in this case $(\partial c/\partial\omega)$ has a block-diagonal structure and that $(\partial c_u/\partial\omega_u) = -I_d$.Thus, $(\partial c/\partial\omega)(\omega^*)$ satisfies (2.24) if and only if $(\partial c_\tau/\partial\omega_\tau)(\omega_\tau^*)$ satisfies (2.24)

In particular, following the discussion of Sturm-Liouville properties in the Cahn-Halliard example, we conclude that for one-dimensional deformations, $\tau, u \in \mathbb{R}^1$, we find co-periodic stability of spatially periodic solutions of (4.21) if and only if $(dc_\tau/d\omega_\tau)(\omega_\tau^*) > 0$ and $dL/d\tau_{\max} > 0$, that is, if period is increasing monotonically with amplitude. This recovers and extends the corresponding results of [44]. The phrasing of the stability condition in terms of $(dc_\tau/d\omega_\tau)$ yields also further insight, we feel, that is not evident in the original phrasing of [44].[14]

For multidimensional deformations $\tau \in \mathbb{R}^d$, we obtain as a necessary and sufficient condition for co-periodic stability that the number of positive eigenvalues of $(\partial c_\tau/\partial\omega_\tau)(\omega_\tau^*)$ is equal to the number of negative eigenvalues of $-\nabla^2\Psi(\tau_x^*)\partial_x^2 + \nabla^2 W(\tau^*)$, considered as an operator on all of $L^2[0,1]_{per}$ (i.e., with no zero-mass restriction). *This result appears to be completely new.*

[14] In our notation, the necessary condition of [44] is effectively $\det(\partial\omega/\partial u_-) \times \det(\partial u_-/\partial c) > 0$, with $\omega = f(u_-)$.

Moreover, though we do not state it in detail here, we obtain a corresponding characterization of stability also for multiply-periodic solutions of the full d-dimensional equations of viscoelasticity with strain-gradient effects, $d = 1, 2, 3$, for which $\tau \in \mathbb{R}^{d\times d}$ and $u \in \mathbb{R}^d$, yielding $d^2 + d$ constraints C_j on constrained minimization problem (2.1), again concerning the mechanical energy

$$\mathcal{E}(\tau, u) := \int_0^1 \Big(\frac{1}{2}|u|^2 + W(\tau) + \Psi(\tau_x)\Big)dx. \tag{4.29}$$

We obtain as a necessary and sufficient condition for co-periodic stability that the number of positive eigenvalues of $(\partial c_\tau/\partial\omega_\tau)(\omega^*_\tau)$, a symmetric $(d^2 + d) \times (d^2 + d)$-dimensional tensor, is equal to the number of negative eigenvalues of $-\nabla^2\Psi(\tau^*_x)\partial_x^2 + \nabla^2 W(\tau^*)$. For discussion of the full equations and their derivation, see for example [5, 13]. *This result, again, appears to be completely new*; indeed, the approach by other (e.g., Evans function) means appears quite forbidding in likely complexity. Again, we collect the results from this subsection in the following proposition.

Proposition 4.9. *Assume that (τ^*, u^*) is a **1D-periodic or multiD–periodic** pattern of* (4.21) *The following assertions hold true:*

(i) *For the case of one-dimensional deformations $\tau \in \mathbb{R}$, the periodic pattern(τ^*, u^*) is stable if and only if*

$$(dc_\tau)/(d\omega_\tau)(\omega^*_\tau) > 0 \text{ and } (dL)/(d\tau_{\max}) > 0;$$

(ii) *For the case of multidimensional deformations $\tau \in \mathbb{R}^d$, the periodic pattern (τ^*, u^*) is stable if and only if the number of positive eigenvalues of $(\partial c_\tau/\partial\omega_\tau)(\omega^*)$ is equal to the number of negative eigenvalues of $-\nabla^2\Psi(\tau^*_x)\partial_x^2 + \nabla^2 W(\tau^*)$, considered as an operator on all of $L^2[0, 1]_{per}$;*

(iii) *For the case of full d-dimensional equations of viscoelasticity with strain-gradient effects, the multiply-periodic pattern (u^*, v^*) is stable if and only if the number of positive eigenvalues of $(\partial c_\tau/\partial\omega_\tau)$, a symmetric $(d^2 + d) \times (d^2 + d)$-dimensional tensor, is equal to the number of negative eigenvalues of $-\nabla^2\Psi(\tau^*_x)\partial_x^2 + \nabla^2 W(\tau^*)$ considered as an operator on all of $L^2[0, 1]_{per}$.*

Remark 4.10. It has been shown by other means in [6, 61] that variational and time-evolutionary stability are equivalent for (4.21); here, we recover these results as a consequence of our general theory. We note that it was shown in [6, 61] also that the spectrum of MA_{ω^*} is real for $\mathrm{Re}\lambda \geq 0$, which is special to the precise structure of the model, and does not hold in general.

Remark 4.11. As in Remark 4.2, we find, integrating the traveling-wave ODE (4.26) over one period, that $\omega^*_\tau = -\nabla W(\tau^*)^a$, where superscript a denotes average over one period, giving the explicit (singly periodic) stability condition

$$d(W'(\tau^*)^a)/dc_\tau < 0 \text{ and } dL/d\tau_{\max} > 0 \text{ (case } d = 1) \tag{4.30}$$

or $\sigma_-\big(\partial(\nabla W(\tau^*)^a)/\partial c_\tau\big) = \sigma_-\big(A_{\omega^*}\big)$ (case $d \geq 1$), where $c_\tau = \int_0^1 \tau^*(x)dx$.

Remark 4.12. On their natural spaces $\mathbf{H} = (H^1 \times L^2)[0, 1]_{\text{per}}$ and $\mathbf{H} = (L^2 \times H^1)[0, 1]_{\text{per}}$, respectively (the spaces on which the second variation is bounded and coercive), (4.23) and (4.9) are not only not C^2, but not even continuous, since τ_x and u enter nonlinearly, but are not controlled in L^∞ by the norm associated with $\mathbf{H}$.

4.4. **Generalized KdV.** It is well-known from the celebrated work of Grillakis, Shatah and Strauss ([18]), that the stability of solitary waves arising in models such as the Klein-Gordon equation or the generalized KdV equation can be studied using variational results similar to our results. We briefly mention here the generalized KdV model and the corresponding functionals. Consider the equation

$$u_t + (\nabla F(u))_x = u_{xxx}. \tag{4.31}$$

Here $u \in \mathbb{R}^m$ and $F : \mathbb{R}^m \to \mathbb{R}$ is a smooth function. This equation can have periodic waves with respect to co-periodic perturbations as shown in [9] or solitary waves on the whole line as discussed in [18]. This equation also fits the general framework (3.3) with

$$\mathcal{E}(u) := \int_0^1 \Big(\frac{1}{2}|u_x|^2 + F(u)\Big)dx \tag{4.32}$$

and

$$J = \partial_x, \qquad \nabla_{L^2}\mathcal{E}(u) = \nabla F(u) - \partial_x^2 u. \tag{4.33}$$

In the case of periodic patterns we can take $x \in [0,1]_{\rm per}$ and consider (4.31) in the energy space $\mathbf{H} = H^1_{\rm per}[0,2]$, with mass constraint $C(u) = (\int_0^1 u(x)dx, \int_0^1 u^2(x)dx)$.

Remark 4.13. It is readily verified that energies (4.2),(4.32) are C^2 on $\mathbf{H} = H^1[0,1]_{\rm per}$ or $H^1(\mathbb{R})$. Moreover, there is a weak existence theory [40] on the energy space $\mathbf{H}$, verifying condition (B2).

4.5. **General conservation laws.** For periodic solutions of general parabolic conservation laws

$$u_t + (f(u))_x = (B(u)u_x)_x,$$

without the assumption of generalized gradient form, Oh and Zumbrun ([44], Thm. 5.9) have obtained by direct Evans function/Stability index techniques the related *necessary stability condition* that $\det\Big((\partial c/\partial\omega)(\omega^*)\Big)$ have a certain sign sgnγ determined by the orientation properties of u^* as a solution of the underlying traveling-wave ODE. The Jacobian $(\partial c/\partial\omega)(\omega^*)$ is connected to the (necessary) stability index condition of Theorem 5.9, [44] through the observation that, in the notation of the reference,

$$\begin{aligned}\det(\partial u_m^*/\partial u_-)\det df(u_-) &= \det(\partial u_m^*/\partial u_-)(\partial u_-/\partial q) = \det(\partial u_m^*/\partial q)\\ &= \det\Big((\partial c/\partial\omega)(\omega^*)\Big),\end{aligned}$$

where $q = \omega^*$ and $u_m^* = c$. This is somewhat analogous to the relation established in [46] between Evans and variational stability conditions in the Hamiltonian solitary wave case; see also [60]. The sign sgnγ may be recognized, by a homotopy argument similar to that used to prove the full index relation, as measuring the parity of the sign of the Morse index of the linearized traveling-wave operator $\Big(B(u^*(x)\partial_x + dB(u^*(x))u_x^*\Big) - df(u^*(x))$ equivalent (by integration) to the restriction to the set of zero-mass functions $\nabla C(u^*)^\perp$ of the full linearized operator

$$\partial_x\Big(B(u^*(x))\partial_x + dB(u^*(x))u_x^* - df(u^*(x))\Big),$$

where $C(u) := \int_0^1 u(x)dx$, thus completing the analogy to the generalized gradient case. Indeed, as this argument reiterates, the stability index itself may be viewed as a mod two version of the classical Sturm–Liouville theory applicable to the non-selfadjoint case; see especially the discussion of [15, 55] from a similar perspective of the general self-adjoint case, for which the zeros counted in the second-order scalar case are replaced by zeros of an appropriate Wronskian.

5. Sideband instability and modulational dichotomies

We now describe some interesting consequences regarding the Whitham modulation equations and modulational stability of periodic waves, in particular, the modulational dichotomies mentioned in the introduction. The Whitham modulation equations, a formal WKB expansion in the high-frequency limit, is known (see [54, 45, 32, 4, 24] and especially [42, 43]) to have a dispersion relation agreeing to lowest order with the spectral expansion of the critical eigenmodes of L_{ω^*} determining low-frequency (sideband) stability. In particular, *well-posedness of the Whitham equation* may be seen to be *necessary for low-frequency modulational stability.*

As critical, or slow, modes are associated to linear order with variations $\partial X/\partial\omega$ and $\partial X/\partial s$ along the manifold of nearby stationary solutions, the equations naturally involve the same quantities already studied in relation to co-periodicity, and this turns out to yield in some situations substantial further information. For details of the derivation of the Whitham equations in the generality considered here, see, for example, [54, 45].

5.1. **Viscoelasticity with strain-gradient effects.** We consider the first order Whitham system for (4.21) is (see [31]):

$$\begin{aligned} \tau_t^a - u_x^a &= 0, \\ u_t^a - \nabla W(\tau)_x^a &= 0, \\ k_t &= 0, \end{aligned} \tag{5.1}$$

where superscript a denotes average over one period, k denotes wave number, or one over period. By the block-triangular structure, we may ignore the trivial third equation and concentrate on the first two. Indeed, this is a common feature of all of our dissipative examples, as the general k-equation is $k_t - (\sigma(\omega,k)k)_x = 0$, where σ denotes speed of the associated periodic traveling wave, which for gradient-type systems is identically zero.

Integrating the traveling-wave ODE (2.4)

$$-(\nabla\Psi(\tau_x))_x + \nabla W(\tau) + \omega = 0,$$

over one period, where ω is the Lagrange multiplier for the associated constrained minimization problem in τ, we find that $\nabla W(\tau)^a = -\omega$, hence (5.1) writes more simply as

$$\begin{aligned} c_t - d_x &= 0, \\ d_t + \omega_x &= 0, \end{aligned} \tag{5.2}$$

where $c = \tau^a$ is the constraint for the minimization problem, $d = u^*$ is a parameter for the minimization problem (irrelevant, by Galillean invariance with respect to u), and $\omega = \omega(c)$ independent of u is determined by the solution structure of the standing-wave problem. This is of exactly the same form as the equations of viscoelasticity themselves, whence the condition for hyperbolicity of the equations is that $\partial\omega/\partial c$ (recall, this is a Hessian for the constrained minimization problem, so symmetric) be *negative definite.*

It has been shown that hyperbolicity of the Whitham system is necessary for sideband stability. This means that periodic waves are sideband stable *only if* $\partial\omega/\partial c < 0$, in which case Corollary 2.9, implies that they are co-periodic stable only if the second variation operator

$$-\nabla^2\Psi(\tau_x^*)\partial_x^2 + \nabla^2 W(\tau^*)$$

is nonnegative on all of $H^1[0,1]_{\rm per}$, not just for perturbations with zero mean. This is false in the scalar case $\tau \in \mathbb{R}$, by Sturm-Liouville theory and the fact that τ_x^* is a zero-eigenfunction that changes sign. Thus, we recover in straightforward fashion the dichotomy result of [44].

Theorem 5.1. *In the scalar case, $\tau, u \in \mathbb{R}$, assume that (τ^*, u^*) is a periodic solution of (4.21). Then (τ^*, u^*) is either co-periodic unstable or else it is sideband unstable: in all cases, modulationally unstable. For systems $\tau, u \in \mathbb{R}^d$, solutions are modulationally unstable unless $A_{\omega^*} = -\nabla^2\Psi(\tau_x^*)\partial_x^2 + \nabla^2 W(\tau^*)$ is nonnegative on all of $H^1[0,1]_{\rm per}^d$.*

This yields for the first time an extension to multidimensional deformations of the dichotomy of [44]. Moreover, it gives new information even in the one-dimensional deformation case, removing a superfluous technical assumption (see Theorem 7.1, [44]) that period T be increasing with respect to amplitude a. This both simplifies the analysis, eliminating the often-difficult problem of determining sgn(dT/da), and answers the question left open in [44] whether there might exist modulationally stable periodic waves with period decreasing with respect to amplitude.

Further, though, as discussed in Section 4.1.1, we cannot say much in the general system case regarding nonnegativity of A_{ω^*}, due to additional structure we can in the present situation say substantially more. Specifically, consider the class of multidimensional deformation solutions obtained by continuation from the special class of *decoupled solutions*, equal to a one-dimensional deformation solution in one of its components, say, (τ_{j_*}, u_{j_*}), and constant in the other components (τ_j, u_j), $j \neq j_*$. This generates a large subclass of the possible multidimensional deformation solutions: in some cases, perhaps, all.

Corollary 5.2. *In the (singly periodic) system case, all periodic solutions of* (4.21) *lying on a branch of transverse solutions extending from a decoupled solution are modulationally unstable.*

Proof. By Sturm-Liouville considerations, decoupled solutions (reducing to the scalar case), satisfy $\sigma_-(A_{\omega^*}) \geq 1$, hence are modulationally unstable. Noting that $\mathrm{Ker}(A_{\omega^*})$ has constant dimension so long as the solution remains transversal, we find that no eigenvalues can pass through the origin and so $\sigma_-(A_{\omega^*}) \equiv 1$ along the entire branch, yielding the result by Theorem 5.1. □

Remark 5.3. Though transversality is difficult to check analytically, it is straightforward to implement as part of a numerical continuation study. From a practical point of view, the main importance of Corollary 5.2 is that waves obtained through numerical continuation from decoupled solutions are unstable at least up to the point that bifurcation first occurs.

Remark 5.4. It is worth noting that the solutions bifurcating from the one-dimensional case include truly multi-dimensional ones and not only one-dimensional solutions in disguise. As noted implicitly in the description (5.1), transverse planar periodic solutions of (4.21) may be locally parametrized, up to x-translates, by the wave number $k = 1/L$, where L is the period, the mean $c \in \mathbb{R}^d$ of τ over one period, and the material velocity $u \equiv$ constant- the latter entering trivially, by invariance of (4.21) under the Galillean transformation $u \to u + c$, c constant. Among these solutions are "disguised" one-dimensional solutions, for which τ is confined to an affine set; see [58] for further classification/discussion. However, a convenient asymmetry in the coefficient $1/\tau_3$ of the parabolic term in u for the standard choice of viscosity (4.22) forces such solutions to lie either in the hyperplane $\tau_3 \equiv 1$ or else along the line $\tau_1 = \cdots = \tau_{d-1} = 0$, so that solutions can be effectively one-dimensional in this sense only if $\tau_3 \equiv 1$, or else $\tau_1 \equiv \tau_2 \equiv 0$. But, these possibilities can be excluded by choosing a τ-mean c for which $c_3 \neq 1$ and $(c_1, \dots, c_{d-1}) \neq (0, \dots, 0)$, of which there are uncountably many.

Remark 5.5. Theorem 5.1 resolves a question left open in [44] whether there might exist modulationally stable waves in the case $dL/d\tau_{\max} < 0$. The reason for the difference in these two sets of results is that the treatment of co-periodic stability in [44] proceeds by a stability index (Evans function) computation, which gives information on the parity of the number of unstable eigenvalues of the linearized operator about the wave and not the actual number as determined here.

5.1.1. *The multiply periodic case.* It is interesting to consider the implications of sideband stability for multiply periodic solutions of the full d-dimensional equations of viscoelasticity with strain-gradient effects, for which the first-order Whitham homogenized system is (ignoring the decoupled k-equation)

$$\begin{aligned} c_t - \nabla_x d &= 0, \\ d_t + \nabla_x \omega &= 0, \end{aligned} \tag{5.3}$$

$x \in \mathbb{R}^d$, with c now a matrix $\in \mathbb{R}^{d\times d}$ and $d \in \mathbb{R}^d$, or $c_{tt} - \nabla_x(\nabla_x \omega(c)) = 0$. For this system, well-posedness (hyperbolicity) is well-known [13] to be equivalent to *rank-one convexity* of $(\partial c/\partial \omega)$, a less-restrictive condition than the convexity condition arising in the one-dimensional case $x \in \mathbb{R}^1$. (Recall, under this notation, $c = \tau^a \in \mathbb{R}^{d\times d}$.) This enforces only d positive eigenvalues in the signature of $(\partial c/\partial \omega)$, with another $d(d-1)$ zero eigenvalues forced by rotational symmetry, leaving $d^2 - d$ possible negative eigenvalues, in principle compatible with variational stability. At the same time, the relation between variational and time-periodic stability is unclear, as the argument of Lemma 3.20 yielding (E1) breaks down in the fully multidimensional case.[15] Thus, it seems possible that one might find *stable* multiply periodic solutions of the equations of viscoelasticity.

5.2. **Cahn–Hilliard systems.** The second-order Whitham modulated system for (4.1) may be seen by a similar (straightforward) derivation as in [54] to be

$$\begin{aligned} u^a_t &= -\partial_x^2(\partial_x^2(u) - \nabla W(u))^a, \\ k_t &= \partial_x(\alpha(u^a,k)\partial_x(u^a) + \beta(u^a,k)\partial_x k). \end{aligned} \tag{5.4}$$

where α, β, since irrelevant for our considerations here, are left unspecified. Integrating the traveling-wave ODE (2.4)

$$-u'' + \nabla W(u) + \omega = 0,$$

over one period, where ω is the Lagrange multiplier for the associated constrained minimization problem, we find that $(\nabla W(u) - u'')^a = \nabla W(u)^a = -\omega$, hence (5.4) writes more simply as

$$\begin{aligned} c_t &= -\partial_x^2(\omega) = -\partial_x((\partial\omega/\partial c)\partial_x c), \\ k_t &= \partial_x(\alpha(c,k)\partial_x c + \beta(c,k)\partial_x k). \end{aligned} \tag{5.5}$$

where $c = u^a$ is the constraint for the minimization problem. This system is well-posed (parabolic), corresponding to sideband stability, only if $\partial\omega/\partial c$ is negative definite, with the same conclusions as in the case of viscoelasticity just considered.

In particular, in the scalar case, recovering a result of [26], we may immediately conclude existence of a dichotomy as in [44], stating that waves are either sideband unstable or co-periodic unstable; in either case, unstable. As in the previous subsection, we obtain also extensions to the system case.

It is worth mentioning that the one-dimensional theory just described extends essentially unchanged to the multiply periodic case, i.e., stationary solutions in $H^1[0,1]^m_{\text{per}}$. Again, there is a single constraint c equal to the mean over $[0,1]^m$, and a single Lagrange multiplier ω, with standing-wave PDE

$$\Delta u - \nabla W(u) = -\omega.$$

In the scalar case, the condition for co-periodic stability is $\partial c/\partial\omega > 0$, by our abstract theory.

On the other hand, the Whitham modulation equations are easily found to be

$$\begin{aligned} c_t &= \Delta\omega = -\nabla_x \cdot ((\partial\omega/\partial c)\nabla_x c), \\ k_t &= \nabla_x \cdot (\alpha(c,k)\nabla_x c + \beta(c,k)\nabla_x k), \end{aligned}$$

$k \in \mathbb{R}^m$, hence are well-posed only if $(\partial\omega/\partial c) \le 0$. We have therefore that multiply periodic waves are either co-periodically unstable or else sideband unstable, a dichotomy similar to that observed for singly periodic scalar waves, implying that multiply-periodic stationary solutions of the scalar Cahn–Hilliard equation are always modulationally unstable under assumption (H3). *So far as we know, this result for multiply-periodic Cahn–Hilliard waves is new* (though see Remark 5.7 below); the connection to sideband stability likewise appears to be a novel addition to the Cahn-Hilliard literature.

In the system case, we obtain a partial analog, similarly as for viscoelasticity, namely, co-periodically stable solutions of (4.1) are necessarily sideband unstable, unless $A_{\omega^*} = -\Delta_x +$

[15] Specifically, $(M + M^*)$ is no longer definite in the u coordinate, a key element in the proof.

$\nabla^2 W(u^*)$ is nonnegative on $H^1[0,1]^m_{per}$. We collect the results of this subsection in the following theorem.

Theorem 5.6. *Assume u^* is a periodic solution of* (4.1)*. Then u^* is either co-periodic unstable or sideband unstable. In the system case, co-periodically stable waves are sideband unstable unless $A_{\omega^*} = -\Delta_x + \nabla^2 W(u^*)$ is nonnegative on $L^2[0,1]^m_{\text{per}}$.*

Remark 5.7. In the scalar, one-dimensional case, we may obtain the result of modulational (*but not necessarily sideband*) instability more directly, as in [20], by the observation that, by Sturm–Liouville considerations, periodic waves cannot be co-periodically stable on a doubled domain $[0, 2T]$, since the unconstrained problem has at least two unstable modes, only one of which may be stabilized by the constraint of constant mass. Recalling that modulational stability is equivalent to co-periodic stability on multiple periods $(0, NT)$ for arbitrary N, we obtain the result. In the two-dimensional case, we may obtain, similarly, co-periodic instability on the multiple domain $(0, NT] \times (0, NT]$, by the Nodal Domain Theorem of Courant [12], using a bit of planar topological reasoning to conclude that there must either be a nodal domain in each cell that is disconnected from the boundary, or else a nodal curve which disconnects two opposing edges of the cell boundary, and thus there must be at least $2N$ nodal domains in $(0, NT] \times (0, NT]$, and thereby $2N - 1$ unstable modes, again yielding modulational instability for $N \geq 2$. This argument is suggestive also in dimensions $m = 3$ and higher, but would require further work to eliminate the possibility that there are only two nodal domains in $(0, NT]^m$, for example a thickened m-dimensional lattice (no longer disconnecting space in dimensions $m > 3$) and its complement.

Remark 5.8. It would be interesting to consider the consequences of our theory in the planar multidimensional case considered by Howard in [27], that is, the implications for transverse instability.

5.3. **Coupled conservative-reaction systems.** For equations (4.8), it is readily seen that the second-order Whitham modulated system is

$$
\begin{aligned}
u^a_t &= \partial_x\Big(b(u)^a \partial_x (F'(u) - v)\Big),\\
k_t &= \partial_x\Big(\alpha(u^a, k)\partial_x(u^a) + \beta(u^a, k)\partial_x k\Big),
\end{aligned}
\tag{5.6}
$$

so that, denoting $c = u^a$, $b(u)^a = \beta(c, k) > 0$, and referring to the traveling-wave equation (4.11), we obtain the equation

$$
\begin{aligned}
c_t &= -(\beta(c,k)\omega_x)_x = -(\beta(c,k)(\partial\omega/\partial c)c_x)_x,\\
k_t &= (\alpha(c,k)c_x + \beta(c,k)k_x)_x,
\end{aligned}
$$

which is well-posed (parabolic) only if $(\partial c/\partial\omega)(\omega_*) < 0$, in contradiction with co-periodic stability. Thus, we find a dichotomy as in the previous cases.

Likewise, for multiply periodic solutions of

$$
\begin{aligned}
u_t &= \nabla_x \cdot \Big(a(u)\nabla_x u - b(u)\nabla_x v\Big),\\
v_t &= \Delta_x v + \delta u + g(v),
\end{aligned}
\tag{5.7}
$$

or stationary solutions in $H^1[0,1]^m_{\text{per}}$, we obtain

$$
\begin{aligned}
c_t &= -\nabla_x \cdot (\beta(c,k)\partial(\omega/\partial c)\nabla_x c),\\
k_t &= \nabla_x \cdot (\alpha(c,k)\nabla_x c + \beta(c,k)\nabla_x k),
\end{aligned}
$$

$k \in \mathbb{R}^m$, where c is equal to the mean over $[0,1]^m$, and ω is the corresponding Lagrange multiplier, again yielding a modulational dichotomy unless A_{ω_*} is not nonnegative on $H^1[0,1]^m_{per}$, We collect the results of this subsection in the following theorem.

Theorem 5.9. *Assume that (u^*, v^*) is a periodic solution of (4.8). Then (u^*, v^*) is either co-periodic unstable or else it is sideband unstable: in all cases, modulationally unstable. Multiply periodic solutions of (5.7) are modulationally unstable unless A_{ω^*} is nonnegative on $H^1[0,1]^m_{\text{per}}$.*

5.4. **Generalized KdV.** Results of [8] for the generalized KdV equation, relating sideband stability to Jacobian determinants of action variables, appear to be an instance of a similar phenomenon in the Hamiltonian case. This would be a an interesting direction for further study.

5.5. **Dichotomies revisited.** The above examples can be understood in a simpler and more unified way through the underlying quasi-gradient structure of the linearized generator $\mathcal{L} = -MA$. Namely, we have only to recall that *sideband instability* is defined as stability for $\xi \in \mathbb{R}$, $|\xi| << 1$ of the small eigenvalues of the Floquet operator

$$\mathcal{L}_\xi := e^{-i\xi x}\mathcal{L}e^{i\xi x} = M_\xi A_\xi \tag{5.8}$$

bifurcating from zero eigenvalues of $\mathcal{L}_0 = \mathcal{L}$, where the ξ subscript indicates conjugation by the multiplication operator $e^{i\xi x}$, or

$$\mathcal{L}_\xi := e^{-i\xi x}\mathcal{L}e^{i\xi x}, \quad M_\xi := e^{-i\xi x}Me^{i\xi x}, \quad A_\xi := e^{-i\xi x}Ae^{i\xi x}.$$

Modulational stability, similarly, is defined as stability for $\xi \in \mathbb{R}$, of $M_\xi A_\xi$.

Note first that A_ξ inherits automatically the self-adjoint property of A, and also the quasi-gradient property

$$\tfrac{1}{2}(M_\xi + M_\xi^*) \geq 0 \text{ for all } |\xi| << 1, \tag{5.9}$$

as seen by the computation $(M_\xi + M_\xi^*) = (M + M^*)_\xi = (M + M^*)_\xi^{1/2}(M + M^*)_\xi^{1/2}$, both of which follow from the adjoint rule

$$(N_\xi)^* = (N^*)_\xi, \tag{5.10}$$

where the $*$ on the lefthand side refers to $L^2[0, L]_{\text{per}}$ and the $*$ on the righthand side refers to $L^2(\mathbb{R})$ adjoint, valid for any operator N that has an $L^2(\mathbb{R})$ adjoint N^*, is L-periodic in the sense that

$$(Nf)(x + L) = \big(Nf(\cdot + L)\big)(x) \tag{5.11}$$

for C^∞ test functions f, and is bounded from $W^{s,\infty}$ to L^∞ for s sufficiently large: in particular, for the periodic-coefficient differential operators considered here in the applications. We note further that, under these assumptions, N_ξ and N_ξ^* take L-periodic functions to L-periodic functions.

The latter statement follows by noting that N_ξ is also invariant under shifts by one period. The former follows by noting that, for u, v periodic, H^s and satisfying

$$u(x + L) = \gamma u(x),\ v(x + L) = \gamma v(x),\ |\gamma| = 1, \tag{5.12}$$

we have that $\langle u, Nv\rangle_{L^2[0,L]}$ is equal to L times the mean of $u \cdot Nv$ over $\mathbb{R}$. This follows by smooth truncation on bigger and bigger domains, with truncation occurring over a single period, and noting that this bounded error goes to zero in the limit. But, then, this is also equal to the mean of $N^*u \cdot v$ over $\mathbb{R}$, by taking the L^2 adjoint for the truncated approximants and arguing as before by continuity. Finally, observe that $\langle e^{-i\xi x}Ne^{i\xi x}u, v\rangle = \langle N\tilde{u}, \tilde{v}\rangle$, where $\tilde{u} = e^{i\xi x}u$ and $\tilde{v} = e^{i\xi x}v$ satisfy the above property (5.12). Thus, it is equal to $\langle \tilde{u}, N^*\tilde{v}\rangle = \langle u, (N^*)_\xi v\rangle$ as claimed. For periodic-coefficient differential operators $p(\partial_x) = \sum_{j=0}^r a_j\partial_x^j$, (5.10), and the property that periodic functions are taken to periodic functions, may be verified directly, using the rule

$$(p(\partial x))_\xi = p(\partial_x + i\xi).$$

Now, assume, as is easily verified in each of the cases considered (but need not always be true), that

$$\ker M_\xi = \emptyset, \tag{F1}$$

so that the corresponding linear evolution equation $(d/dt)Y = -MAY$ is unconstrained (has no associated conservation laws). Then, *for M self-adjoint*, we obtain the correspondence

$$\sigma_-(A) \le \sigma_-(A_\xi) = n_-(M_\xi A_\xi), \tag{5.13}$$ [16]

where n_- is the number of negative eigenvalues counted by algebraic multiplicity, as may be seen by the similarity transformation

$$M_\xi A_\xi \to M_\xi^{-1/2} M_\xi A_\xi M_\xi^{1/2} = M_\xi^{1/2} A_\xi M_\xi^{1/2}.$$

Here, we are assuming sufficient regularity in coefficients that eigenfunctions remain in proper spaces upon application of $M_\xi^{\pm 1/2}$ and also implicitly that spectrum is discrete, properties again easily verifiable in each of the cases previously considered. More generally, the result carries over to the case (D1') provided $(M_\xi + M_\xi) > 0$; see Remark 3.23.

Thus, we find immediately that modulational stability is violated *unless A_ξ is unconditionally stable* for $|\xi| << 1$, which, in the limit $\xi \to 0$, gives unconditional (neutral) stability of A. Moreover, again invoking continuity of spectra with respect to ξ, we find that, assuming co-periodic stability, or conditional stability of A, that the negative eigenvalues of $M_\xi A_\xi$ must be small eigenvalues bifurcating from zero-eigenvalues of MA, which, by the property (E1) as guaranteed by Lemma 3.20, are the only neutral (i.e., zero real part) eigenvalues of MA.

This argument shows that, assuming the property (5.13), first, modulational stability requires unconstrained stability of A, and, second, if co-periodic stability holds, then sideband stability requires unconstrained stability of A. The second observation may be recognized as exactly the modulational dichotomy just proved case-by-case, the first as a generalization of the one obtained by nodal domain consideration in Remark 5.7.

Remark 5.10. We obtain by this argument modulational dichotomies for Cahn–Hilliard and coupled conservative–reaction diffusion equations. On the other hand, we don't (quite) obtain a result for the viscoelastic case, for which M is not self-adjoint, and so our simple argument for (5.13) does not apply. We can recover this however by a "vanishing viscosity" argument, noting for $\varepsilon > 0$ that $n_-(M(\varepsilon)_\xi A_\xi) = \sigma_-(A_\varepsilon)$ for $M(\varepsilon) = M + \varepsilon \mathrm{Id}$ and, by property (D1'), $n_0(M(\varepsilon)_\xi A_\xi) \equiv n_0(M(\varepsilon)_\xi A_\xi) = \sigma_0(A_\xi) = 0$. Mimicking the proof of Lemma 4.8, we find, similarly, that the center subspace of $M_\xi A_\xi$ consists of the zero-subspace of $M_\xi A_\xi$, which is likewise empty. Thus, no zeros cross (or reach) the imaginary axis during the homotopy and we obtain the result in the limit as $\varepsilon \to 0$, for any $\xi \neq 0$ sufficiently small that A_ξ does not have a kernel.

Remark 5.11. The above argument clarifies somewhat the mechanism behind observed modulational dichotomy results; namely, the origins of such dichotomies are a priori knowledge of modulational instability (encoded, through considerations as above, or as in Remark 5.7, in unconstrained instability of A_{ω_*}) plus the central property (E1) that the center subspace of MA_{ω_*} consists entirely of $\ker MA_{\omega_*}$.

6. Discussion

We conclude by briefly discussing how our results connect to other stability studies.

6.1. Relation to the Evans function.

First, we note the following connection to the Melnikov integral and the Evans function:

Remark 6.1. If $r = 1$, the quantity

$$(\partial c/\partial \omega) = \langle \nabla c, (\partial X/\partial \omega) \rangle = -\langle A_\omega (\partial X/\partial \omega), (\partial X/\partial \omega) \rangle$$

[16] The strict inequality $\sigma_-(A) < \sigma_-(A_\xi)$ is possible in the cases we consider, due to bifurcation of $\ker A$.

is a Melnikov integral involving variations in ω along the manifold of nearby stationary solutions, and as pointed out in [46, 60] and elsewhere should correspond to the first nonvanishing derivative of an associated Evans function.

Of course, we don't need this connection, since we have an if and only if condition for asymptotic stability under our assumption (E1), which is perhaps an analogy in the dissipative context to the [18] assumption in the Hamiltonian setting that J be one to one (the difference being that (E1) seems to be satisfied for all the systems we know of/are interested in, while J one-to-one does not hold for (gKdV) and other primary examples).

6.2. **Localized structures: problems on the whole line.** The results of [26, 47, 61] on exponential instability of solitary, or "pulse-type" exponentially localized spikes solutions suggest another dichotomy different from the one we proved in Section 5: if essential spectrum of the linearization $L = -MA_{\omega^*}$ along the spike is good, then point spectrum must be bad. The two dichotomies are apparently different, but do resemble each other — for, co-periodic stability is related to point spectrum, sideband to essential spectrum (thinking of the homoclinic limit [16]).[17]

Moreover, we can recover (slightly weakened versions of) the above mentioned results on instability of spikes from [26, 47, 61] using our results on variational stability of periodic solutions. Below we will illustrate the main ideas in the specific case of the viscoelasticity model (4.21) with $d = 1$. First, we assume that the essential spectrum of the linearization along the spike is stable, otherwise the instability is trivial. We note that the quasi-gradient structure implies that it is enough to prove variational instability in order to obtain (neutral) time-evolutionary instability. In [61, 58] it was shown that for the viscoelasticity with strain-gradient effects model (4.21) the two types of instability are actually equivalent.

We suspect this is true for a larger class of models, but choose not to pursue this here. Rather, we just notice that instability of A_{ω^*}, *without constraint* implies already an exponentially localized initial data for the evolution problem on the whole line for which $\mathcal{E}$ is nondecreasing in time and initially strictly less that $\mathcal{E}(X^*)$, whereas the manifold of nearby solitary wave solutions, consisting entirely of translates of X^*, has energy $\equiv \mathcal{E}(X^*)$. Thus, by continuity, $X(t)$ cannot converge in H^2 to the set of translates of X^*, else the energies would also converge; this shows that the standard notion (see [61, 58]) of $L^1 \cap H^s \to H^s$ orbital asymptotic stability cannot hold for $s \geq 2$.

Next, we remark that any homoclinic spike solution can be approximated by a sequence of periodic patterns having large period $2T$. This is true for any system that has a Hamiltonian structure, in particular the models (4.1), (4.8) and (4.21). Then, we use the convergence results for the point spectrum from [16, 50] to prove the (above type of neutral) instability of the homoclinic spike solution by showing that the approximating periodic pattern is unstable for any $T > T_* > 0$.

Averaging in (4.26), one readily checks that

$$\omega_\tau = -\frac{1}{2T}\int_{-T}^{T} W'(\tau^*(x))\,dx \to -W'(\tau^\infty), \quad \text{as} \quad T \to \infty. \tag{6.1}$$

Similarly, using the definition of the constraint function, we have that

$$\frac{c_\tau}{2T} = \frac{1}{2T}\int_{-T}^{T} \tau^*(x)\,dx \to \tau^\infty, \quad \text{as} \quad T \to \infty. \tag{6.2}$$

[17] In view of the discussion of Section 5.5, the link may be just that modulation and the large-period limit are related through small-ξ, or "long-wave" response, both having the effect of "removing" constraints.

From (6.1) and (6.2) we conclude that

$$T\frac{\partial \omega_\tau}{\partial c_\tau} \to -W''(\tau^\infty), \quad \text{as} \quad T \to \infty.^{18} \tag{6.3}$$

Since τ^* is a solution of (4.26) having finite limit at $\pm\infty$ we infer that $W''(\tau^\infty) > 0$. Our claim follows immediately from Proposition 4.9 (i) and (6.3), yielding a sequence of eigenvalues $\nu_T > 0$ converging as $T \to \infty$ to an eigenvalue ν of A_{ω^*}. With further work, exploiting the results in [50], one may show, provided the limiting operator A_{ω^*} has a spectral gap at $\nu = 0$, that the associated eigenfunctions f_T have uniform exponential decay at $\pm\infty$, hence extract a subsequence converging to an eigenfunction f of A_{ω^*}; moreover, the same uniform exponential decay allows us to conclude that f is both orthogonal to the translational eigenfunction X^*_x and satisfies the linearized constraint, with eigenvalue $\nu \geq 0$ corresponding to "neutral" (i.e., nonstrict) orbital instability as claimed. Provided that $\mathrm{Ker}(A_{\omega^*})$ is spanned by the translational mode X^*_x, we obtain in fact *strict variational instability* $\nu > 0$.

On the other hand, we have, more simply, just by Sturm–Liouville considerations, that A is variationally unstable without constraints, without the above construction. Either argument leads to the conclusion that asymptotic orbital stability cannot hold. If one can verify further that the derivative of the standard Evans function does not vanish at $\lambda = 0$, then one could go further (by property (E1)) to conclude strict, or exponential instability as shown in [61] (and analogously for reaction diffusion–conservation law spikes in [48]). Such a result would then yield by [16] the known result of co-periodic instability for sufficiently large period, completing the circle of arguments.

Remark 6.2. As noted in [61], stability of front-type solutions may also be studied variationally for viscoelasticity with strain-gradient effects. It would be interesting to try to phrase this entirely in terms of the structure (3.10); in particular, we suspect that the similar result of [38] in the context of chemotaxis is another face of the same basic mechanism.

Remark 6.3. In the cases considered in [26, 47, 61], the existence problem

is scalar second order with spectral gap at $\nu = 0$, whence it is easily by dimensionality seen that X^*_x is the only element (up to constant multiple) of $\mathrm{Ker}(A_{\omega^*})$, hence we indeed recover strict constrained variational instability from the limiting periodic argument above. However, we do not see a correspondingly easy way to see a priori that the derivative of the Evans function does not vanish at the origin, so for the moment obtain by this argument only *neutral instability* for the time-evolutionary problem.

Remark 6.4. The above instability arguments (including those of [26, 47, 61]) rely on spectral gap of A_{ω^*} at the origin, $\nu = 0$, and the associated property of exponential convergence of X^* to its endstate as $x \to \pm\infty$. In the critical case of an algebraically-decaying homoclinic solution, it has been shown for the Keller–Segel equation that stability holds, with algebraic decay rate [11, 7].

6.3. **Coupled conservative–reaction diffusion spikes on bounded domains.** As pointed out in [48], coupled conservation-reaction diffusion spikes have been numerically observed [14, 22, 33, 34], suggesting that they may sometimes be stable on finite domains. As we pointed out earlier, the stability with periodic boundary conditions is equivalent to stability with Neumann boundary conditions, since odd eigenfunctions are automatically stable, due to the fact that the translational derivative possesses only two sign changes in a domain of minimal period. Assuming monotonicity of the period, one concludes that patterns are stable whenever $dc/d\omega > 0$. Now letting the period tend to infinity, there are just a few possibilities for a periodic solution: convergence to a homoclinic solution, convergence to a heteroclinic solution, or unbounded amplitude. In fact, all

[18] As the reverese limit $dc/d\omega \to \infty$ indicates, solutions with different multipliers ω are infinitely far apart in the whole-line problem, and do not play a role; in particular, $\ker MA = \mathrm{Span}\{X^*_x\}$ is one-dimensional.

scenarios occur in relevant circumstances. We already discussed the case of the spike limit, in which all periodic solutions are unstable. In the Cahn-Hilliard example with, say, a cubic nonlinearity $W'(u) = -u + u^3$, one finds stable solutions periodic solutions that converge to a heteroclinic loop as the period tends to infinity. In fact, the energy is coercive in this case, so that there necessarily exists a constrained minimizer for any prescribed mass. For masses $|C(u)| < 1/\sqrt{3}$, homogeneous equilibria are unstable so that coercivity of the energy enforces the existence of stable equilibria for all periods, which, given a priori bounds from the energy, therefore necessarily converge to a heteroclinic loop (or a pair of layers/kinks). Similar considerations apply to the coupled conservative reaction-diffusion context when we have a globally coercive energy. In the case of chemotaxis, this global coercivity fails. Also, the nonlinear oscillator that describes periodic solutions possesses only two equilibria, which excludes heteroclinic loops. In the large-wavelength limit, one finds spike-like solutions which *do not* converge to a homoclinic solutions, but diverge along a family of spikes to infinity as the period goes to infinity, consistent with our previous discussion.

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