Generalized Projection Operators In Banach
Spaces And Orthogonal Decomposition Theorems

Yakov I. Alber
Department of Mathematics
The Technion-Israel Institute of Technology
32000 Haifa, Israel

ABSTRACT: It is well known that an arbitrary element $x$ of a Hilbert space $H$
admits the Beppo Levi decomposition in the shape of sum of two mutually orthogonal
(metric) projections $P_Mx$ and $P_M^\perp x$ of this element on a subspace $M$ and its orthogonal
complement $M^\perp$, i.e.,

$$x = P_Mx + P_M^\perp x,$$

where 

$$(P_M^\perp x, v) = 0, \ \forall v \in M.$$ 

Here $(x, y)$ denotes the inner product of $x$ and $y$ in $H$.

This representation shows that $P_Mx$ is the best approximation of $x$ among all of the
elements of the subspace $M$. This is a basis for many deep results in various areas of math-
ematics such as the geometry of spaces, functional analysis, differential equations, approxi-
mation, and optimization.

Moreau (1962) has extended this result to convex closed cones in Hilbert spaces in the
form:

$$x = P_Kx + P_{K^0}x, \ (P_Kx, P_{K^0}x) = 0,$$

where $K$ is an original cone and $K^0$ is its polar cone in $H$.

In the actual lecture, we present similar results in uniformly convex and uniformly
smooth Banach spaces. For that we must use not only metric projection operators, but also
the generalized projection operators introduced by author in 1992. We also state a number
of applications of the obtained decomposition theorems.