NOTES ON TODORCEVIC’S ERICE LECTURES ON FORCING
WITH A COHERENT SUSLIN TREE

PAUL B. LARSON

1. Part I

1.1. The P-ideal dichotomy. The P-ideal dichotomy is the statement that whenever \( I \) is a P-ideal on a set \( X \), either \( X \) is a countable union of sets orthogonal to \( I \) (i.e., intersecting no member of \( I \) infinitely), or there is an uncountable subset of \( X \) whose countable subsets are all in \( I \). The statement is not weakened when we assume that \( I \) consists of countable sets, which we do here.

First we review a proper forcing which forces an instance of the P-ideal dichotomy. Let \( X \) and \( I \) be as above, and assume that \( X \) is not a countable union of sets orthogonal to \( I \). Fix \( a = (\kappa, H(\kappa))^{+} \). For each countable elementary submodel \( M \) of \( H(\kappa) \) with \( X \) and \( I \) in \( M \), fix an element \( a_M \) of \( I \) which contains mod finite all members of \( M \cap I \). Let \( P \) be the partial order whose conditions \( p \) are pairs \((M_p, Y_p)\), where \( M_p \) is a finite \( \in \)-chain of countable elementary submodels of \( H(\kappa) \) with \( X \) and \( I \) as members, and \( Y_p \) is a finite \( M_p \)-separated (i.e., for any two members of \( Y_p \) there is an element of \( M_p \) that has one as an element and not the other) subset of \( X \) such that for each \( y \in Y_p \) and each \( M \in M_p \), if \( y \in M \) then \( y \in a_M \), and if \( y \notin M \) then \( y \) is not in any set in \( M \) orthogonal to \( I \). The order is inclusion on both coordinates.

Now suppose that \( p \) is a condition, and \( N \) is a countable elementary submodel of \( H((2^{(P)})^{+}) \) with \( P \) and \( p \) as elements. Let \( p' \) be the condition \((M_p \cup \{N\cap H(\kappa)\}, Y_p)\). We want to see that \( p' \) is \((P, N)\)-generic. So let \( D \) be a dense subset of \( P \) in \( N \) and let \( r \) be a condition below \( p' \). We may assume that \( r \in D \). Let \( M_0 \) be the largest model in \( M_r \cap N \).

Arguing in \( N \) and identifying finite subsets of \( X \) with their increasing enumeration in terms of some wellordering of \( X \) in all models of \( M_r \), let \( T \) be the tree of finite increasing sequences \( t \) from \( X \) such that

- all members of \( t \) are greater than all members of \( Y_r \cap N \),
- no member of \( t \) is in any set in \( M_0 \) orthogonal to \( I \),
- there is an extension \( Z \) of \((Y_r \cap N) \cup t \) of length \(|Y_r|\) for which there is some condition \( q \in D \) with \( Y_q = Z \).

Note that \( Y_r \setminus N \) is in \( T \). Now thin \( T \) (iteratively removing as few nodes as possible) to a tree \( T' \) such that for each node \( t \) of \( T' \) of length less than \(|Y_r \setminus N|\) (including the emptyset), the set of \( x \in X \) such that \( t \setminus \langle x \rangle \in T' \) is not orthogonal to \( I \). This thinning takes \(|Y_r \setminus N| \) many rounds starting, one for each non-terminal level of the tree, proceeding from the top down. Note that \( Y_r \setminus N \) is still in \( T' \), since for

Date: December 14, 2012.

The author is supported in part by NSF grant DMS-0401603. Most of the details of this note were worked out in discussion with István Juhasz, Justin Moore and Stevo Todorcevic, at the Mittag-Leffler Institute, in the fall of 2009.

1
each proper initial segment $t$ of $Y \setminus N$, $t$ is in some elementary submodel $M$ of $H(\kappa)$, and the next element of $Y$ above the maximum of $t$ is not in any set in $M$ orthogonal to $I$.

Now we can choose a cofinal branch through $T'$ consisting of elements of $N$, with the property that the elements of the branch are all in $a_M$ for all $M \in M_p \setminus N$. To see this, note that at each point of our construction the set of possible extensions in $T'$ must contain an infinite element of $I$, and all but finitely many of the members of $I$ will be in $\bigcap_{M \in M_p \setminus N} a_M$.

This completes the proof.

1.2. \textbf{PFA($S$) and the P-ideal dichotomy.} Now suppose that $S$ is a coherent Suslin tree, $\lambda$ is a cardinal and $I$ is an $S$-name for a P-ideal on $\lambda$ such that $\lambda$ is not a countable union of sets orthogonal to $I$. Again, let $\kappa$ be $(\lambda^{<\lambda})^+$. For each countable elementary submodel $M$ of $H(\kappa)$ with $S$ and $I$ as members, we choose a name $\dot{a}_M$ for a countable subset of $\lambda$ such that the members of $S_{\omega_1 \cap M}$ (where $S_\alpha$ denotes the $\alpha$th level of $S$) decide $\dot{a}_M$, and the realization of $\dot{a}_M$ is forced to

- contain mod finite all members of the realization of $\dot{I}_M$.
- be contained mod finite in some member of $I$ containing mod finite all members of the realization of $\dot{I}_M$.

where $\dot{I}_M$ is the name for the realization of all the names in $M$ for members of $I$. We can find such a name by filling an appropriate $(\omega, \omega)$-gap corresponding to each member of $S_{\omega_1 \cap M}$. Since we assume that $\dot{I}$ is a name for an ideal containing all finite subsets of $\lambda$, $\dot{a}_M$ is in fact a name for a member of the realization of $\dot{I}$.

For each such $M$, let $\xi_M$ be the canonical (nice) name for the least element of $\lambda$ not in any subset of $\lambda$ orthogonal to $\dot{I}$ realized by a name in $M$.

We now define the forcing $P$. A condition in $p$ is a function whose domain is a finite $\in$-chain $M_p$ of countable elementary submodels of $H(\kappa)$ with $S$ and $I$ as members, and range contained in $S$, such that for each $M \in M_p$, $p(M)$ is not in $M$ but is in all members of $M_p \setminus M$, and that $p(M)$ decides the value of $\xi_M$ (note that $p(M)$ will also decide the value of $\dot{a}_M$, though this is less important). The function $p$ must have the further property that if $M, N$ are in $M_p$ and $p(N) < p(M)$, then $p(M)$ forces that $\xi_N \in \dot{a}_M$.

Now suppose that $p_0$ is a condition in $P$, and $N$ is a countable elementary submodel of $H((2^{<\lambda})^+) \cap P$ with $p_0$ as members. Let $p_1$ be the condition $p_0 \cup \{ (N \cap H(\kappa), t') \}$, where $t'$ is any element of $S \setminus N$. Now let $s_0$ be any element of $S_N \cap \omega_1 \setminus M$. We need to see that $(p_1, s_0)$ is $(P \times S, N)$-generic.

Let $(r, s_1)$ be an element of $P \times S$ below $(p_1, s_0)$. We may assume that $(r, s_1) \in D$, and that the height of $s_1$ is greater than the height of any member of the range of $r$. Fix $\gamma_0 \in \omega_1 \cap N$ such that no member of the range of $r$ disagrees with $s_0$ at any point in the interval $[\gamma_0, \omega_1 \cap N)$. Enumerate the models of the domain of $r$ (as ordered by the $\in$-relation) as $(Q_i : i < |r|)$.

For any condition $p \in P$, let $\Xi_p$ be the function with the same domain as $p$ where $\Xi_p(Q)$ is the value of $\xi_Q$ as decided by $p(Q)$.

For each $t \in T$, let $T_t$ be the tree of consisting of all initial segments of increasing sequences $\epsilon$ from $X$ which are the ranges of $\Xi^p_{(r \cap N)}$, for some $p \in P$ end-extending $(r \cap N)$ such that $(p, t) \in D$, $|p| = |r|$ and
for each $i < |r|$, if $M$ is the $i$th element of the domain of $p$, then $p(M)$ agrees with $r(Q_i)$ up to $\gamma_0$, and $p(M)$ agrees with $t$ after $\gamma_0$ if and only if $r(Q_i)$ agrees with $s_1$ after $\gamma_0$.

Let $a$ be the set of $i < |r|$ such that $r(Q_i)$ does not disagree with $s_1$ on ordinals greater than or equal to $\gamma_0$.

Since $D$ is closed under strengthening the right coordinate, $T_i \subseteq T_i'$ whenever $t \geq S t'$.

For each $t \in S$, thin $T_i$ to a tree $T_i'$ (iteratively removing as few nodes as possible, level by level) such that for each $\sigma \in T_i$ (including the empty sequence),

- if $|r \cap N| + |\sigma| + 1 \in a$,
- $B_{\sigma}$ is the set of immediate successors of $\sigma$ in $T_i'$,
then $B$ is forced by the union of $t$ beyond $\gamma_0$ with $r(Q_{|r \cap N| + |\sigma| + 1})|_{\gamma_0}$ to have infinite intersection with some countable set $C$ forced by this condition to be in $\check{I}$ (which since this union is $M$-generic is the same as saying that the union does not force $B$ to be orthogonal to $\check{I}$).

Claim. The range of $\Xi_{\check{r} \setminus N}$ is in $T_s$.

Proof: For each $t \in S$, let $T_i^0 = T_i$. For each ordinal $j < |r \setminus N|$ and each $t \in S$, $T_i^{j+1}$ is formed from $T_i^j$ by thinning removing those sequences from $T_i^j$ of length $|r \setminus N| - j - 1$ whose set of immediate successors is not sufficiently large. It suffices then to fix $j < |r \setminus N|$, to suppose that the range of $\Xi_{\check{r} \setminus N}$ is in $T_i^j$ and show that it is in $T_i^{j+1}$. To do this, let $i = |r \setminus N| - j - 1$, and let $\sigma_i$ be the first $i$ member of the range of $\Xi_{\check{r} \setminus N}$.

Let $U$ be the set of $t \in S$ such that $\sigma_i \in T_i^j$. Then $U \in Q_{i+1}$.

For each $t \in U$, let $B_{\sigma_i}^j$ be the set of immediate successors of $\sigma_i$ in $T_i^j$.

If there exist $t \geq S t'$ in $S \cap Q_{i+1}$ below $s_1$ such that $t'$ forces $B_{\sigma_i}^j$ not to be orthogonal to $I$, then we are done. Otherwise, there is a name in $Q_{i+1}$ for the union of the sets $B_{\sigma_i}^j$ along the generic branch, and this set must be forced by some initial segment of $s_1$ in $Q_{i+1}$ to be orthogonal as it is an increasing union of uncountably many orthogonal sets. But $s_1$ forces that $\Xi_{\check{r}(Q_{i+1})}$ is not in this set, and $\Xi_{\check{r}(Q_{i+1})} \in B_{\sigma_i}^n$, giving a contradiction. This concludes the proof of the claim.

Then $T_s$ has height $|r|$, so the set of $s \in S$ extending $s_1|_{\gamma_0}$ for which $T_s$ has height $|r|$ contains $s_1$, so we can find such a $s_2$ in $N$ which is an initial segment of $s_1$. Then we can find a condition of size $|r|$ in $N \cap T_{s_2}$ such that the corresponding $\xi$’s are in the required realizations of the names $a_M$, minus their finite errors. We do this by finding in $N$ a branch $p_2$ (i.e., $p_2$ is the set of left-coordinates of the branch) through $T_{s_2}$ with the property that for each $M \in M_{p_2}$ and each $Q \in M_r \setminus N$, if $p_2(M) < r(Q)$, then $\xi_M$ as decided by $p_2(M)$ is in the set $\check{a}_Q$ as decided by $r(Q)$. Note that as we do this, if $i < |r|$ and $r(Q_i)$ disagrees with $s_1$ above $\gamma_0$, then the same will be true for the $i$th level of $p_2$, so the hypotheses of the above implication will not be satisfied. In the other case, the set of values $\xi_M$ for potential models $M$ at the $i$th level (according to $T_{s_2}$) is forced by the union of $s_2$ beyond $\gamma_0$ with $r(Q_i)|_{\gamma_0}$ to have infinite intersection with some countable set $C$ forced by this condition to be in $\check{I}$. Then for each $Q \in r \setminus N$ such that $r(Q)$ agrees with $r(Q_i)$ up to $\gamma_0$, $\check{a}_N$ (as decided by $s_1[\gamma_0, N \cap \omega_1] \cup r(Q_i)|_{\gamma_0}$ contains all but finitely much
of $C$, so there is some member of $C \cap B$ which is in all of these sets. Choose the $i$th model $M$ of $P_2$ so that the realization of $\xi_M$ is such a member. This completes the proof that $(p_1, s_0)$ is $(P \times S, N)$-generic.

Finally, let us suppose that $\langle M_\alpha : \alpha < \omega_1 \rangle$ is a generic sequence for $P$, with a corresponding function $p$ whose domain is this sequence and whose range is contained in $S$. The set of conditions $p(M_\alpha)$ is somewhere dense in $S$, and any branch through $S$ below this condition will force that the realizations of the names $\xi_{M_\alpha}$ for which $p(M_\alpha)$ is in the generic branch will be an uncountable set whose countable subsets are all in the realization of $I$. Since we could carry out this entire argument below any node of $S$, a dense set of nodes in $S$ force the existence of such an uncountable set and this completes the proof that under PFA($S$) the P-ideal dichotomy holds after forcing with $S$. 

Department of Mathematics, Miami University, Oxford, Ohio 45056, USA
E-mail address: larsonpb@muohio.edu
URL: http://www.users.muohio.edu/larsonpb/