1 The stationary tower

1.1 Definition. Let $X$ be a nonempty set. A set $c \subseteq P(X)$ is club in $P(X)$ if there is a function $f : X^{<\omega} \to X$ for which $c$ is the set of $A \subseteq X$ closed under $f$. Given a cardinal $\kappa \leq |X|$, $c$ is club in $[X]^\kappa$ (or in $[X]^{<\kappa}$) if $c$ is the set of $A \subseteq X$ closed under $f$. A set $a \subseteq P(X)$ is stationary in $P(X)$ if it intersects every club subset of $P(X)$, and stationary in $[X]^\kappa$ (or $[X]^{<\kappa}$) if it intersects every $c$ which is club in $[X]^\kappa$ (or $[X]^{<\kappa}$).

If $c$ is club in $P(X)$, then $\bigcup c = X$, so we can simply say that $c$ is club if it is club in $\bigcup c$. Similarly, if $a$ is stationary in $P(X)$, then $\bigcup a = X$, so we can simply say that $a$ is stationary if it is stationary in $P(\bigcup a)$.

The two following facts were discussed previously.

1.2 Remark. For any first order structure on a nonempty set $X$, a Skolem function for the structure induces a club of elementary substructures.

Lemma 1.3 (The projection lemma for stationary sets). Suppose that $X \subseteq Y$ are nonempty sets, and $\kappa \leq |X|$ is a cardinal.

1. If $a$ is stationary in $P(Y)$, then $\{B \cap X \mid B \in a\}$ is stationary in $P(X)$.

2. If $a$ is stationary in $P(X)$, then $\{B \subseteq Y \mid B \cap X \in a\}$ is stationary in $P(Y)$.

3. If $a$ is stationary in $[X]^\kappa$, then $\{B \in [Y]^\kappa \mid B \cap X \in a\}$ is stationary in $[Y]^\kappa$.

4. If $a$ is stationary in $[X]^{<\kappa}$, then $\{B \in [Y]^{<\kappa} \mid B \cap X \in a\}$ is stationary in $[Y]^{<\kappa}$.

Lemma 1.4 (Normality for stationary sets). Suppose that $a$ is a stationary set, and that $f : a \to \bigcup a$ is such that $f(X) \in X$ for all $X \in a$. Then there is a $z \in \bigcup a$ such that $f^{-1}([z])$ is stationary.

Proof. Otherwise, for each $z \in \bigcup a$ there is a function $g_z : (\bigcup a)^{<\omega} \to \bigcup a$ such that no $X \in f^{-1}([z])$ is closed under $g_z$. Fix a function $h : (\bigcup a)^{<\omega} \to \bigcup a$ with the property that $h(z, y) = g_z(y)$ for all $z \in \bigcup a$ and $y \in (\bigcup a)^{<\omega}$. Then if $X \in a$ is closed under $h$ we get a contradiction by considering $z = f(X)$. \qed
1.5 Definition. We define the following order on stationary sets: \( a \leq b \) if \( \bigcup b \subseteq \bigcup a \), and

\[ \{ X \cap \bigcup b \mid X \in a \} \subseteq b. \]

Note that if two stationary sets \( a, b \) are compatible in this order, they have a greatest lower bound: \( \{ X \subseteq (\bigcup a) \cup (\bigcup b) \mid X \cap \bigcup a \in a \land X \cap \bigcup b \in b \} \).

1.6 Definition (The stationary tower). Given a limit ordinal \( \alpha > \omega \), we let \( P_{\leq \alpha} \) the (full tower) be the restriction to the stationary sets in \( V_{\alpha} \) of the order given in Definition 1.5. Given a cardinal \( \kappa < |V_\alpha| \), we let \( Q_{<\kappa}^\alpha \) (the size-\( \kappa \) tower, or countable tower in the case where \( \kappa = \aleph_0 \)) be the restriction of \( P_{<\alpha} \) to those stationary sets \( a \) for which \( a \subseteq [a]^{\aleph_0} \). We write \( Q_\alpha \) for \( Q_{<\aleph_0}^\alpha \).

Suppose that \( \alpha > \omega \) is a limit ordinal, and that \( G \subseteq P_{<\alpha} \) (or \( Q_{<\kappa}^\alpha \), for some cardinal \( \kappa < |V_\alpha| \)), we form \( \text{Ult}(V,G) \) as follows. Elements of the generic ultrapower are represented by functions \( f: a \to V \), where \( a \in G \) and \( f \in V \). Given such functions \( f \) and \( g \) and a relation \( R \in \{ \in, = \} \), we say that \( [f]_GR[g] \) if and only if \( \{ X \subseteq (\bigcup a) \cup (\bigcup b) \mid X \cap \bigcup a \in a \land X \cap \bigcup b \in b \land f(X \cap \bigcup a)Rg(X \cap \bigcup b) \} \in G \). As usual, we identify the wellfounded part of \( \text{Ult}(V,G) \) with its Mostowski collapse, and denote \( \text{Ult}(V,G) \) with a single capital letter if it is wellfounded.

1.7 Remark. Suppose that \( c \in P_{<\alpha} \) is club or that \( c \in Q_{<\kappa}^\alpha \) is the intersection of some club subset of \( \bigcup c \) with \( [\bigcup c]^{\aleph_0} \). Then \( c \) is an element of every generic filter for its corresponding partial order. The generic ultrapower \( \text{Ult}(V,G) \) could alternately defined using functions of the form \( f: c \to V \) for \( c \)'s of this type.

We have seen the proof of the following lemma several times already.

Lemma 1.8. Suppose that \( G \) is as in the previous paragraph. For each stationary \( a \) and each set \( x \), let \( e^x_a: a \to \{ x \} \) be the corresponding constant function. Then for every set \( x \), \( [e^x_a]_G = [e^x_b]_G \) for all \( a, b \in G \). The map \( j: V \to \text{Ult}(V,G) \) sending each \( x \) to the common value of \( [e^x_a]_G \) for all \( a \in G \) is elementary. \( x \)

1.9 Remark. Forcing with \( P_{<\alpha} \) (or \( Q_{<\kappa}^\alpha \)) adds a \( V \)-ultrafilter \( G_X = G \cap \mathcal{P}(\mathcal{P}(X)) \) on \( \mathcal{P}(X) \), for each suitable \( X \in V_{\alpha} \) (note that these \( G_X \) are not necessarily \( V \)-generic for the (suitable version of the) partial order of stationary subsets of \( \mathcal{P}(X) \) modulo nonstationarity). Let \( j_X: V \to \text{Ult}(V,G_X) \) be the corresponding elementary embedding for each \( X \). Then if \( X \subseteq Y \) are in \( V_{\alpha} \), there are elementary embeddings \( k_{X,Y}: \text{Ult}(V,G_X) \to \text{Ult}(V,G_Y) \) defined by setting \( k_{X,Y}(f)_{G_X} = f^Y_{G_Y}, \) where \( f^Y \) is defined by setting \( f^Y(Z) = f(Z \cap X) \). Similarly, there are elementary embeddings \( k_{X,\infty}: \text{Ult}(V,G_X) \to \text{Ult}(V,G) \) defined by setting \( k_{X,\infty}(f)_{G_X} = f_{G} \). The word tower refers to the fact that the stationary tower ultrapower can be seen as a direct limit of generic ultrapowers via the \( G_X \)'s.

1.10 Remark. For any nonempty set \( X \), \( \{ X \} \) is stationary. In any suitable \( P_{<\alpha} \) or \( Q_{<\kappa}^\alpha \), the condition \( \{ X \} \) is in the generic filter \( G \) if and only if the filter \( G_X \) as defined in the previous remark is principal.
Lemma 1.11. Let $\alpha > \omega$ be a limit ordinal, and let $G \subseteq \mathcal{P}_{<\alpha}$ (or $\mathcal{Q}_{<\alpha}^\kappa$) be a V-generic filter.

1. For each set $X \in V_\alpha$, and any $a \in G$ with $X \subseteq \bigcup a$ the function $i_x$ on a defined by setting $i_x(Y) = X \cap Y$ represents $j[X]$ in Ult($V, G$).

2. The wellfounded part of Ult($V, G$) contains $V_\alpha$.

3. For each $a \in \mathcal{P}_{<\alpha}$ (or $\mathcal{Q}_{<\alpha}^\kappa$), $a \in G$ if and only if $j(\bigcup a) \in j(a)$.

4. For each $\beta < \alpha$, $G \cap V_\beta \in$ Ult($V, G$).

Proof. The first part follows from normality. The second part follows from the fact that for each transitive set $X$, the transitive collapse of $j[X]$ is $X$. For the third part, note that $j(\bigcup a)$ is represented by the identity function $i$ on $\mathcal{P}(\bigcup a)$ (or on $[\bigcup a]^{<\kappa}$) and $j(a)$ is represented by the constant function $c_a$ from $\mathcal{P}(\bigcup a)$ (or $[\bigcup a]^{<\kappa}$) to $\{a\}$. Then in the case of $\mathcal{P}_{<\alpha}$, $j(\bigcup a) \in j(a)$ if and only if

$$\{X \subseteq \bigcup a \mid i(X) \in c_a(a)\} \in G,$$

which holds if and only if $a \in G$. The case of $\mathcal{Q}_{<\alpha}^\kappa$ is essentially the same.

For the last part, note that for each $\beta < \alpha$, $j|V_\beta$ is in Ult($V, G$), and $G \cap V_\beta$ is equal to the set of $X \in V_\beta$ for which $j[X] \in j(X)$.

1.12 Remark. The previous lemma shows that each element $X$ of $V_\alpha$ is represented by the function (on any suitable domain) which maps each set $Y$ to the transitive collapse of $X \cap Y$. Similarly, it shows that each set of the form $[f]_G$ also has the form $j(f)(j(\bigcup a))$, for $j$ the generic elementary embedding.

1.13 Exercise. Let $\alpha > \omega$ be a limit ordinal, let $G \subseteq \mathcal{P}_{<\alpha}$ (or $\mathcal{Q}_{<\alpha}^\kappa$) be a V-generic filter, and let $j: V \rightarrow$ Ult($V, G$) be the corresponding embedding. Then, for any pair of ordinals $\beta \leq \delta$ below $\alpha$, and any relation $R \in \{\leq, =, \geq\}$, $j(\beta) R \delta$ if and only if

$$\{X \in \delta \mid \beta R \text{ot}(X \cap \delta)\} \in G,$$

(or $\{X \in [\beta \cup \kappa]^\kappa \mid \beta R \text{ot}(X \cap \delta)\} \in G$). Show that in the case of a generic embedding $j$ induced by $\mathcal{Q}_{<\alpha}^\kappa$, $j(\kappa^\omega)$ is at least $\alpha$.

1.14 Exercise. If $\lambda$ is a cardinal of uncountable cofinality, then $\lambda$ is a stationary subset of $\mathcal{P}(\bigcup \lambda)$ (but club only in the case $\lambda = \omega_1$). If $\kappa$ and $\lambda$ are infinite cardinals, then $\lambda \cap [\bigcup \lambda]^\kappa$ is a stationary subset of $[\bigcup \lambda]^\kappa$ only in the case $\lambda = \kappa^\omega$.

1.15 Remark. Even in the case $\lambda = \omega_2$ and $\kappa = \omega_1$, $[\bigcup \lambda]^\kappa \setminus \lambda$ can be stationary, i.e., if Chang’s Conjecture, the statement that $[\omega_2]^{\omega_1}$ (the set of subsets of $\omega_2$ of ordertype $\omega_1$) is stationary, holds. By Exercise 1.13, the condition $[\omega_2]^{\omega_1}$ (if stationary) forces (in $\mathcal{P}_{<\alpha}$ or $\mathcal{Q}_{<\alpha}^\kappa$) that $j(\omega_1) = \omega_2$.

Lemma 1.16. If $\lambda$ is a cardinal of uncountable cofinality, $\alpha > \lambda$ and $G \subseteq \mathcal{P}_{<\alpha}$ is a V-generic filter, then $\lambda \in G$ if and only if the critical point of the induced
elementary embedding is \( \lambda \). Similarly, if \( \kappa \) is a cardinal, \( \lambda \leq \kappa^+ \) is a cardinal, \( \alpha > \kappa^+ \) is a limit ordinal and \( G \subseteq Q_{\leq \alpha}^c \) is a \( V \)-generic filter, then
\[
\{ X \in [\kappa^+]^\kappa | X \cap \lambda \in \lambda \} \in G
\]
if and only if \( \lambda \) is the critical point of the induced elementary embedding.

**Proof.** For the first part, \( \lambda \in G \) if and only if \( j[\bigcup \lambda] \in j(\lambda) \), which holds if and only if \( j[\lambda] \) is an ordinal below \( j(\lambda) \), which holds if and only if \( \lambda \) is the critical point of \( j \). Similarly, for the second part, \( \{ X \in [\kappa^+]^\kappa | X \cap \lambda \in \lambda \} \in G \) if and only if \( j[\kappa^+] \cap j(\lambda) \in j(\lambda) \), which holds if and only if \( j[\kappa^+] \cap j(\lambda) \) is an ordinal below \( j(\lambda) \), which holds if and only if \( \lambda \) is the critical point of \( j \). \( \square \)

Recall that the cardinals \( \beth_n \) are defined by \( \beth_0 = \aleph_0 \), \( \beth_{\alpha+1} = 2^{\beth_\alpha} \) and \( \beth_\beta = \sup_{\alpha < \beta} \beth_\alpha \) when \( \beta \) is a limit ordinal. For any ordinal \( \alpha \), \( |V_{\beth+\alpha}| = \beth_\alpha \). A \( \beth \)-fixed point is a cardinal \( \kappa \) for which \( \beth_\kappa = \kappa \), i.e., for which \( |V_\kappa| = \kappa \).

**Lemma 1.17.** Suppose that \( \kappa \) is a \( \beth \)-fixed point, and that \( G \subseteq P_{<\kappa} \) is a \( V \)-generic filter. Then there is an uncountable regular cardinal \( \lambda < \kappa \) such that \( \lambda \in G \).

**Proof.** Let \( a \in V_\kappa \) be a stationary set. By genericity, it suffices to find a regular cardinal \( \lambda < \kappa \) such that \( \lambda \) is compatible with \( a \). Applying the fact that \( \kappa \) is a \( \beth \)-fixed point, choose a cardinal \( \gamma \in (|\bigcup a|, \kappa) \). Let
\[
b = \{ X \in V_{\gamma+} : |X| < \gamma \wedge \gamma \in X \wedge X \cap \bigcup a \in a \}.
\]
Then \( b \) is stationary, and \( b \subseteq a \). Define the function \( f : b \to (\gamma + 1) \) by letting \( f(X) \) be the unique ordinal \( \beta \in X \) for which \( X \cap \beta \in \beta \). Each value taken by \( f \) is a regular cardinal, and there is a cardinal \( \lambda \) such that \( c = \{ X \in b | f(X) = \lambda \} \) is stationary. Then \( c \leq b \) and \( c \leq \lambda \). \( \square \)

**1.18 Exercise.** Show that if \( \alpha > \omega \) is a limit ordinal, then the generic elementary embedding induced by \( P_{<\alpha} \) always has a critical point.

**1.19 Exercise.** Suppose that \( \kappa \) is a cardinal, \( \alpha > \kappa^+ \) is a limit ordinal and \( G \subseteq Q_{<\alpha}^c \) is a \( V \)-generic filter. Show that there exists a \( \lambda \leq \kappa^+ \) such that \( \{ X \in [\kappa^+]^\kappa | X \cap \lambda \in \lambda \} \in G \).

**1.20 Exercise.** Suppose that \( \kappa \) is a cardinal and \( \alpha > \kappa^+ \) is a limit ordinal. Show that forcing with \( Q_{<\alpha}^c \) below the condition \( \kappa^+ \setminus \kappa \) adds surjections from \( \kappa \) onto each \( V_\beta \) with \( \beta < \alpha \).

**1.21 Exercise.** Suppose that \( V=L \) and that \( \alpha > \omega_1 \) is a limit ordinal. Show that there is a function \( f : \omega_1 \to \omega_1 \) such that \( [f]_G > \beta \) for all \( \beta < \alpha \), whenever \( G \subseteq Q_{<\alpha}^c \) is \( V \)-generic and \( j \) is the associated embedding (see Remark 1.22 of the previous set of notes). Note that \( j(\omega_1) > [f]_G \).
1.22 Remark. More generally, whenever we force over \( L \) with any forcing of the form \( \mathbb{P}_{<\alpha} \) or \( \mathbb{Q}^{\kappa}_{<\alpha} \), if the critical point of the associated embedding \( j \) is a successor cardinal \( \gamma^+ \), then \( j(\gamma^+) \) is illfounded, as \( \text{Ult}(V,G) \) will satisfy \( V=L \) but will contain new subsets of \( \gamma \).

1.23 Extra Credit. Suppose that \( \alpha > \omega \) is a limit ordinal but not a \( \beth \)-fixed point. Must there be a condition \( a \in \mathbb{P}_{<\alpha} \) forcing that the critical point of the induced elementary embedding is at least \( \alpha \)?

2 Completely Jónsson cardinals

2.1 Definition. A strongly inaccessible cardinal \( \kappa \) is completely Jónsson if for all \( a \in \mathbb{P}_{<\kappa} \),

\[
\{ X \subseteq V_\kappa \mid X \cap \bigcup a \in \mathcal{A} \land |X \cap \kappa| = \kappa \}
\]

is stationary.

2.2 Remark. For any ordinal \( \alpha \) and any \( a \in \mathbb{P}_{<\alpha} \), the set

\[
\{ X \subseteq V_\kappa \mid X \cap \bigcup a \in \mathcal{A} \land |X \cap \kappa| \leq \bigcup a \}
\]

is stationary. If \( \kappa \) is completely Jónsson, then for any cardinal \( \lambda \) such that \( |\bigcup a| \leq \lambda \leq \kappa \), the set

\[
\{ X \subseteq V_\kappa \mid X \cap \bigcup a \in \mathcal{A} \land |X \cap \kappa| = \lambda \}
\]

is stationary.

Note that the statement “\( \kappa \) is completely Jónsson” is computed in \( V_{\kappa+1} \).

2.3 Exercise. Show that if we removed from the definition of completely Jónsson the requirement that \( \kappa \) be strongly inaccessible, but required it to be uncountable, it would still follow that \( \kappa \) is a strong limit (indeed, a \( \beth \)-fixed point), but not that it has uncountable cofinality.

The argument for the following is based on the end-extension property of elementary submodels with measurable cardinals (Theorem 1.20 of the previous set of notes).

Theorem 2.4. Measurable cardinals are completely Jónsson, and are limits of completely Jónsson cardinals.

The following follows from Exercise 1.13 and genericity.

Theorem 2.5. If \( \kappa \) is a limit of completely Jónsson cardinals, and \( j \) is a generic elementary embedding derived from forcing with \( \mathbb{P}_{<\kappa} \), then \( j(\lambda) = \lambda \) for cofinally many \( \lambda < \kappa \). If \( \kappa \) is regular then it follows that \( j(\kappa) = \kappa \).
2.6 Remark. In contrast to Lemma 1.11, let us see that if $\kappa$ is a limit of completely Jónsson cardinals, and if $G \subseteq \mathbb{P}_{<\kappa}$ is a $V$-generic filter, then $V_\kappa$ is not a member of $\text{Ult}(V,G)$. Supposing otherwise, there is for some $\alpha < \kappa$ a stationary $\alpha \subseteq \mathcal{P}(V_\alpha)$ and a function $f$ with domain $\alpha$ forced by $a$ to represent $V_\kappa$. We may assume that for each $X \in a$, $f(X)$ is a transitive structure satisfying “there are cofinally many Jónsson cardinals.” Increasing $\alpha$ if necessary, we may assume that $\alpha$ is completely Jónsson and that $\text{ot}(X \cap \alpha) = \alpha$ and $\alpha + 1 \subset f(X)$ hold for all $X \in a$.

At least one of the two following sets is stationary.

- $b_1$, the set of $X \in a$ such that, letting $\beta$ be the least completely Jónsson cardinal of $f(X)$ above $\alpha$, $|f(X) \cap \beta| = \alpha$.

- $b_2$, the set of $X \in a$ such that, letting $\beta$ be the least completely Jónsson cardinal of $f(X)$ above $\alpha$, $|f(X) \cap \beta| > \alpha$.

Let $\beta$ be the least completely Jónsson cardinal above $\alpha$. If $b_1$ is stationary, then so is $c_1$, the set of $Y \prec V_{\beta + 1}$ such that $Y \cap V_\alpha \in b_1$ and $|Y \cap \beta| > \alpha$. If $b_2$ is stationary, then so is $c_2$, the set of $Y \prec V_{\beta + 1}$ such that $Y \cap V_\alpha \in b_2$ and $|Y \cap \beta| = \alpha$.

The function $h$ on $\mathcal{P}(V_{\beta + 1})$ sending $Y$ to its transitive collapse represents $V_{\beta + 1}$. We have then for a (relative) club of $Y \subseteq V_{\beta + 1}$ with $Y \cap \bigcup a \in a$ that the transitive collapse of $Y$ is a rank initial segment of $f(Y \cap V_\alpha)$.

However, if $b_1$ is stationary, then $c_1$ contradicts this, as for each $Y \in c_1$, letting $\bar{Y}$ be the transitive collapse of $Y$ and letting $\beta$ be the least completely Jónsson cardinal of $\bar{Y}$ above $\alpha$, $|\bar{Y} \cap \beta| > \alpha$. Similarly, if $b_2$ is stationary, then $c_2$ gives a contradiction, as for each $Y \in c_2$, letting $\bar{Y}$ be the transitive collapse of $Y$ and letting $\beta$ be the least completely Jónsson cardinal of $\bar{Y}$ above $\alpha$, $|\bar{Y} \cap \beta| = \alpha$.

2.7 Exercise. Prove that if $\delta$ is a limit of completely Jónsson cardinals, then $\mathbb{P}_{<\delta}$ is not $\delta$-c.c.. (Hint: what are the possibilities for $f(\omega_1)$?)

2.8 Example. If $\kappa$ and $\delta$ are cardinals, $\alpha > \kappa$ is a limit ordinal and there exist $\delta$ many measurable cardinals in the interval $(\kappa, \alpha)$, then $\mathbb{Q}_{<\alpha}$ is not $\delta$-c.c.. To see this, let $I$ be a set of $\delta$ many measurable cardinals in the interval $(\kappa, \delta)$ such that no member of $I$ is a limit of members of $I$. For each $\lambda$ in $I$, fix a stationary set $A_\lambda$ consisting of points whose cofinality is the same as that of $\kappa$. By using our usual end-extension trick for measurable cardinals (Theorem 1.20 of the previous set of notes), one can show for each regular cardinal $\rho$ in $(\kappa, \delta)$, that for stationarily many elementary submodels $X$ of $V_\rho$ of cardinality $\kappa$, $\text{sup}(X \cap \lambda) \in A_\lambda$ for each $\lambda$ in $I \cap X$. To make this work for a fixed $\lambda$, end-extend $\lambda$ many times. By the stationarity of $A_\lambda$, some supremum was in the desired $A_\lambda$; then take a hull of some cofinal sequence in this sup of cardinality $\kappa$.

Now, if each $A_\lambda$ were also costationary on the same cofinality, for each $\lambda$ in $I$ the set $b_\lambda$ of $X \prec V_{\lambda +}$ of cardinality $\kappa$ for which $\lambda$ is least in $X \cap I$ with
sup(\(X \cap \lambda\)) not in \(A\) is also stationary, by the same argument. Then the \(b_\lambda\)’s form an antichain.

2.9 Exercise. Let \(\alpha > \omega\) be a limit ordinal, let \(\mathcal{D}\) be a subset of \(P_{<\alpha}\), and let \(\beta < \alpha\) be such that \(\mathcal{D} \subseteq V_\beta\). Let \(a\) be the set of \(X \subseteq V_\beta^{+1}\) of cardinality less than \(|V_\beta|\) such that \(\mathcal{D} \in X\) and, for all \(d \in \mathcal{D}\), \(X \cap \bigcup d \notin d\). Then \(a\) is stationary if and only if \(\mathcal{D}\) is not predense in \(P_{<\alpha}\), and, if \(a\) is stationary, it is incompatible with each element of \(\mathcal{D}\). Show that the same construction works for each partial order of the form \(Q_\kappa\), requiring instead that \(|X| = \kappa\).

2.10 Exercise. Let \(\delta\) be a strongly inaccessible cardinal, let \(\{a_\alpha : \alpha < \delta\}\) be an antichain in \(P_{<\delta}\), and let \(C \subseteq \delta\) be a club set of limit ordinals. For each \(\alpha < \delta\), let \(b_\alpha = \{X \subseteq V_\gamma \mid X \cap \bigcup a_\alpha \in a_\alpha\}\), for \(\gamma\) minimal such that \(X \subseteq V_\gamma\). Then each \(a_\alpha\) is equivalent to \(b_\alpha\) as a condition in \(P_{<\delta}\), so \(B = \{b_\alpha : \alpha < \delta\}\) is an antichain. However, if \(\kappa \in \delta \setminus C\), then \(B \cap P_{<\kappa}\) is not predense in \(P_{<\kappa}\). (Hint: Let \(\beta\) be maximal element of \(C\) below \(\kappa\) and use the previous exercise.) Show that the same argument works for partial orders of the form \(Q_\delta^\kappa\) with the appropriate changes.

3 Wellfoundedness

We say that a set \(Y\) end-extends a set \(X\) if \(X \subseteq Y\) and \(X = Y \cap V_\beta\), for \(\beta\) least such that \(X \subseteq V_\beta\).

3.1 Definition. Let \(\kappa\) be a \(\beth\)-fixed point, and let \(D\) be a subset of \(P_{<\kappa}\). We let \(sp(D)\) be the set of \(X \prec V_\kappa^{+1}\) of cardinality less than \(\kappa\) with \(D \in X\) for which there exists a \(Y \prec V_{\kappa+1}\) satisfying the following.

- \(X \subseteq Y\);
- \(Y \cap V_\kappa\) end-extends \(X \cap V_\kappa\);
- for some \(d \in D \cap Y\), \(Y \cap (\bigcup d) \notin d\).

We say that \(D\) is semi-proper if \(sp(D)\) is club in \([V_\kappa]^{<\kappa}\).

Note that \(Q_\kappa^\beta \subseteq P_{<\kappa}\), and we can assume that \(|Y| = |X|\), so we do not need a separate definition of semi-properness for \(Q_\kappa^\beta\).

3.2 Exercise. Show that if \(\alpha > \omega\) is not a \(\beth\)-fixed point, and \(P_{<\alpha}\) contains an antichain not contained in \(P_{<\beta}\) for any \(\beta < \alpha\), then there is a predense \(D \subseteq P_{<\alpha}\) which is not semi-proper.

3.3 Exercise. Let \(\kappa\) be a \(\beth\)-fixed point. Show that every semi-proper subset of \(P_{<\kappa}\) is predense.

3.4 Exercise. Let \(\kappa\) be a \(\beth\)-fixed point. Show that every predense subset of \(P_{<\kappa}\) in \(V_\kappa\) is semi-proper (with \(X = Y\)).
The following is the stationary tower version of Lemma 4.10 from the first set of notes.

**Lemma 3.5.** Suppose that \( \kappa \) is a \( \square \)-fixed point, and let \( D \) be a subset of \( P_{<\kappa} \).
The following are equivalent.

- \( D \) is semi-proper;
- For any regular cardinal \( \lambda > \kappa \), and any \( X \prec V_\lambda \) with \( \kappa, D \in X \) and \( |X| < \kappa \), there is a \( Y \prec V_\lambda \) such that
  - \( X \subseteq Y \);
  - \( Y \cap V_\kappa \) end-extends \( X \cap V_\kappa \);
  - \( Y \cap (\bigcup d) \in d \) for some \( d \in Y \cap D \).

**Proof.** The reverse direction follows from upwards projection of stationarity. For the forward direction, fix such an \( X \) as given, and let \( W \prec V_{\kappa+1} \) be such that the following hold.

- \( X \cap V_{\kappa+1} \subseteq W \);
- \( W \cap V_\kappa \) end-extends \( X \cap V_\kappa \);
- for some \( d \in D \cap W \), \( W \cap (\bigcup d) \in d \).

Now let \( Y \) be the set of values \( f(a) \), for \( f \) a function in \( X \) with domain \( V_\kappa \), and \( a \in W \cap V_\kappa \). Then \( Y \) is as desired. To show that \( Y \cap V_\kappa \) end-extends \( X \cap V_\kappa \), note that every \( f \colon V_\kappa \to V_\kappa \) in \( X \) is in \( X \cap V_{\kappa+1} \) and thus in \( W \).

**3.6 Exercise.** Verify the previous lemma in the case where we let \( \lambda \) be \( \kappa + \omega \).

**3.7 Exercise.** Let \( \alpha \) be a limit ordinal of uncountable cofinality, and let \( C \subseteq \alpha \) be club. Let \( D \) be the set of elements \( a \in P_\alpha \) such that \( X \cap C \) is cofinal in \( X \cap \text{Ord} \) for every \( X \in a \). Show that \( D \) is predense, and that \( \kappa \in C \) for any \( \kappa \) for which \( D \cap P_\kappa \) is semi-proper. Show that the same holds for towers of the form \( Q_{<\alpha}^\lambda \).

**Lemma 3.8.** Suppose that \( \kappa \) is a strongly inaccessible cardinal and \( \gamma \) is a cardinal below \( \kappa \) such that whenever \( \{ D_\alpha : \alpha < \gamma \} \) are predense subsets of \( P_{<\kappa} \), there exists a \( \square \)-fixed point \( \lambda \) such that each \( D_\alpha \cap V_\lambda \) is semi-proper. Then whenever \( G \subseteq P_{<\kappa} \) is a \( V \)-generic filter, \( \text{Ult}(V,G) \) is closed under sequences of length \( \gamma \).

**Proof.** Let \( \tau_\alpha \) (\( \alpha < \gamma \)) be names for elements of \( \text{Ult}(V,G) \). For each \( \alpha < \gamma \), let \( D_\alpha \) be the set of \( b \in P_{<\kappa} \) for which there exists a function \( f^b_\alpha \) with domain \( b \) such that \( b \) forces \( \tau_\alpha \) to be the element of \( \text{Ult}(V,U) \) represented by \( f_\alpha^b \). Then each \( D_\alpha \) is predense. Fix \( \alpha \in P_{<\kappa} \), and, applying Exercise 3.7, fix \( \lambda \) as in the statement of the lemma with \( a \in V_\lambda \). Let \( \delta > \lambda \) be a regular cardinal. Let \( c \) be the set of \( X \prec V_\delta \) for which

- \( \lambda \in X \);
• $X \cap \bigcup a \in a$;
• for each $\alpha \in X \cap \gamma$ there exists a $b \in D_\alpha \cap X$ such that $X \cap \bigcup b \in b$.

Applying Lemma 3.5, we have that $c$ is stationary, and $c \leq a$. For each $\alpha \in \gamma$, choose a function $h_\alpha : c \rightarrow V_\lambda$ such that for each $X \in C$ and each $\alpha \in X \cap \gamma$, $h_\alpha(X) \in D_\alpha \cap X$ and $X \cap \bigcup h_\alpha(X) \in h_\alpha(X)$. Define a function $g$ on $c$ by letting each $g(X)$ be the function with domain $\ot(X \cap \gamma)$, such that whenever $\alpha \in X \cap \gamma$ and $\ot(X \cap \alpha) = \beta$, $g(X)(\beta) = f^{h_\alpha(X)}(\beta)$. Then $g$ represents a function with domain $\gamma$, and it suffices to see that for each $\alpha \prec \gamma$, $c$ forces that the $\alpha$-th member of the sequence represented by $g$ is equal to the realization of $\tau_\alpha$. To see that this holds, fix $\alpha$, and fix a condition $d \leq c$. We may assume that $\alpha \in Y$ for all $Y \in d$, and by strengthening $d$ if necessary (via normality) that $h_\alpha(Y \cap V_\lambda)$ is the same value $b$ for all $Y \in d$. Since $X \cap \bigcup b \in b$ for all $X \in d$, $b \leq d$, so $d$ forces that the realization of $\tau_\alpha$ is represented by $f^b_\alpha$. Now, the $\alpha$-th member of the sequence represented by $g$ is represented by the function $g_\alpha$ on $c$ for which $g_\alpha(Y \cap V_\lambda) = f^b_\alpha(Y \cap V_\lambda)$ for all $Y \in d$, and therefore that the $\alpha$-th member of the sequence represented by $g$ is equal to the realization of $\tau_\alpha$. \hfill \Box

3.9 Exercise. Prove Lemma 3.8 for towers of the form $Q^\chi_{< \kappa}$ and $\gamma \leq \chi$.

Theorem 3.10. Suppose that $\delta$ is a Woodin cardinal, and let $D_\alpha$ ($\alpha < \delta$) be predense subsets of $P_{< \delta}$. Then there is a measurable cardinal $\lambda < \delta$ such that $D_\alpha \cap V_\lambda$ is semi-proper for each $\alpha < \lambda$.

Proof. Let $f : \delta \rightarrow \delta$ be a function such that

• for each $\alpha < \delta$, $f(\alpha)$ is a $\mathbb{Z}$-fixed point and each $D_\beta$ ($\beta < f(\alpha)$) is predense in $P_{< f(\alpha)}$;
• for each $\mathbb{Z}$-fixed point $\alpha < \delta$ and each $\beta < \alpha$, if $D_\beta \cap V_\alpha$ is not semi-proper, then there is an element of $D_\beta \cap V_{f(\alpha)}$ compatible with $[V_{\alpha+1}]^{< \alpha} \setminus \sp(D_\beta \cap V_\alpha)$.

Applying Theorem 5.19 from the first set of notes, fix $\lambda < \delta$ and an elementary embedding $j : V \rightarrow M$ with critical point $\lambda$ such that

• $j(f)(\lambda) = f(\lambda)$;
• $V_{f(\lambda)+\omega} \subseteq M$;
• $j(D_\alpha) \cap V_{f(\lambda)} = D_\alpha \cap V_{f(\lambda)}$ for all $\alpha < \lambda$;
• $M$ is closed under sequences of length $\lambda$;
• $j(\delta) = \delta$. 

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We want to see that each $D_\alpha \cap V_\lambda$ is semi-proper. Towards this end, fix an $\alpha < \lambda$. Let $a = [V_{\lambda+1}]^{<\lambda} \setminus \text{sp}(D_\alpha \cap V_\lambda)$. If $D_\alpha \cap V_\lambda$ is not semi-proper, then this holds in $M$ also, and there is a condition $b \in j(D_\alpha) \cap V_{f(\lambda)}$ compatible with $a$ in $M$. Then $b \in D_\alpha$, and $a$ and $b$ are compatible in $V$ also.

Let $X$ be an elementary submodel of $V_\delta$ with $X \cap a \in a$, $X \cap \bigcup b \in b$ and the sets $D_\alpha$, $a$, $j(V_{\lambda+1})$, $j(V_{\lambda+1})$ all in $X$. Since $\bigcup a = V_{\lambda+1}$ and each element of $a$ has cardinality less than $\lambda$, $X \cap V_{\lambda+1}$ has cardinality less than $\lambda$. We have then that

- $j(X \cap V_{\lambda+1}) = j(X \cap V_{\lambda+1})$;
- $j(X \cap V_{\lambda+1}) \in j(a)$;
- $j(X \cap V_{\lambda+1}) \notin j(\text{sp}(D_\alpha \cap V_\lambda))$;
- $j(X \cap V_{\lambda+1}) \in M$.

Let $\leq^*$ be a wellordering of $j(V_{\lambda+1})$ in both $M$ and $X$. Since $j(V_{\lambda+1}) \subseteq X$, $j(X \cap V_{\lambda+1}) \subseteq X$. Let $Y$ be the Skolem closure in $j(V_{\lambda+1})$ according to $\leq^*$ of

$$\{a, b\} \cup j(X \cap V_{\lambda+1}) \cup (X \cap (\bigcup a \cup \bigcup b))$$

Since these set are all in $M$, $Y$ is in $M$, and since they are all subsets of $X$, $Y \subseteq X$. We derive a contradiction by showing that $Y$ witnesses in $M$ that

$$j(X \cap V_{\lambda+1}) \in j(\text{sp}(D_\alpha \cap V_\lambda))$$

Since the critical point of $j$ is $\lambda$, and $Y \subseteq X$, $Y$ end-extends $j(X \cap V_{\lambda+1})$ below $j(\lambda)$. We have already that $j(X \cap V_{\lambda+1}) \subseteq Y$. Finally, we have that $b \in Y$ and $X \cap \bigcup b \in b$. Since $j(\lambda) > f(\lambda)$, $b \in j(D_\alpha \cap V_\lambda)$. Therefore, $b \in Y \cap j(D_\alpha)$ and $Y \cap \bigcup b \in b$, finishing the proof.

**Corollary 3.11.** Suppose that $\delta$ is a Woodin cardinal, $G \subseteq \mathbb{P}_{<\delta}$ is a $V$-generic filter and $j: V \rightarrow M$ is the corresponding elementary embedding. Then $j(\delta) = \delta$, $M$ is closed under sequences of length less than $\delta$ and $V_\delta^M = V_\delta^{V[G]}$.

**Proof.** That $M$ is closed under sequences of length less than $\delta$ follows from Lemma 3.8 and Theorem 3.10. That $j(\delta) = \delta$ follows from the fact that $\delta$ is a limit of measurable cardinals, and thus a limit of completely Jönsson cardinals. It follows then that $\delta$ is strongly inaccessible in $M$. Since $M$ is closed under sequences of length less than $\delta$ in $V[G]$, one can prove by induction on $\alpha < \delta$ that $V_\alpha^M = V_\alpha^{V[G]}$ for all $\alpha < \delta$.

**3.12 Exercise.** Prove that if $\delta$ is a Woodin cardinal, and $A_\alpha$ ($\alpha < \delta$) are antichains in $\mathbb{P}_{<\delta}$, then for densely many $b \in \mathbb{P}_{<\delta}$, the set of $a \in A_\alpha$ compatible with $b$ has cardinality less than $\delta$, for each $\alpha < \delta$.

**3.13 Exercise.** Prove Theorem 3.10 for $\mathbb{Q}^\chi_{<\delta}$ and $D_\alpha$ ($\alpha < \chi$), for any cardinal $\chi < \delta$. It follows then that generic ultrapower is closed under sequences of length $\chi$. Prove the corresponding version of Exercise 3.12 for $\chi$ many antichains. Conclude then that $j(\chi^+) = \delta$, so $V_\delta^M = V_\delta^{V[G]}$. 

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It is an open question whether the image of the ordinals under the generic embedding induced by $\mathcal{Q}_{<\delta}$ (for $\delta$ a Woodin cardinal) is independent of the generic filter, or whether it even can be.

3.14 Definition. Given cardinals $\kappa$ and $\lambda$, $\lambda$ is $\lambda$-supercompact if there exists an elementary embedding $j: V \rightarrow M$ with critical point $\kappa$ such that $j(\kappa) > \lambda$ and $M$ is closed under sequences of length $\lambda$.

3.15 Exercise. Prove that if $\kappa$ is $2^\kappa$-supercompact, then every predense subset of $\mathbb{P}_{\leq \kappa}$ (or $\mathcal{Q}_{<\lambda}$, for any cardinal $\lambda < \kappa$) is semi-proper. (Hint: Fix an embedding $j: V \rightarrow M$ witnessing that $\kappa$ is $2^\kappa$ semi-proper, and fix a regular cardinal $\eta > j(\kappa)$. Suppose that some predense $D \subseteq \mathbb{P}_{<\kappa}$ is not semi-proper, and let $a = [V_{\kappa+1}]_{<\kappa} \setminus \text{sp}(D)$. Find an $b \in j(D)$ compatible with $a$ in $j(\mathbb{P}_{<\kappa})$ and an elementary submodel $Y \prec V^M_{\eta}$ in $M$ with $Y \cap \bigcup a \in a$, $Y \cap \bigcup b \in b$ and $\{a, b, j|V_{\kappa+1}\} \in M$. Let $Y = X \cap V_{\kappa+1}$. Show that $j(Y) \in j(a)$ and that $X \cap j(V_{\kappa+1})$ contradicts this.

Exercise 3.15 holds for strongly compact cardinals as well.

4 Forcing applications

4.1 Remark. It is a classical forcing fact that if $\mathbb{B}$ and $\mathbb{C}$ are complete Boolean algebras, and there exists a $\mathbb{B}$-name $\sigma$ for a $V$-generic filter for $\mathbb{C}$ with the property that for each $A \in \mathbb{C}$, $[A \in \sigma] \neq 0_\mathbb{B}$, then there is a $\mathbb{C}$-name $\tau$ for a complete Boolean algebra such that $\mathbb{B}$ is forcing-equivalent to $\mathbb{C} \ast \tau$. To see this, let $\pi: \mathbb{C} \rightarrow \mathbb{B}$ be the embedding defined by letting $\pi(A) = [A \in \sigma]$. Let $\tau$ be a $\mathbb{C}$-name for the Boolean algebra formed by taking the quotient of $\mathbb{B}$ by the $\pi$-image of the complement of the generic filter. That is, let $\tau$ be such that whenever $G$ is $V$-generic for $\mathbb{C}$, $\tau_G$ is the set of equivalence classes of the relation $\sim$ on $\mathbb{B}$ defined by setting $E \sim F$ if and only if $E \cap F \leq \pi(A)$ for some $A \in \mathbb{C} \setminus G$. Then conditions in $\mathbb{C} \ast \tau$ can be represented as pairs $(A, \rho_E)$, where $A \in \mathbb{C}$, $E \in \mathbb{B}$, $E \cap \pi(A') \neq 0_\mathbb{B}$ for all $A' \subseteq A$ in $\mathbb{C}$ and $\rho_E$ is a $\mathbb{C}$-name for the $\sim$-class of $E$. We can map such pairs into $\mathbb{B}$ by setting $\nu(A, \rho_E) = E \cap \pi(A)$.

4.2 Exercise. Show that the embedding $\nu$ defined in Remark 4.1 maps $V$-generic filters for $\mathbb{C} \ast \tau$ to $V$-generic filters for $\mathbb{B}$, and that $\nu^{-1}$ does the reverse.

4.3 Remark. If $P$ and $Q$ are partial orders, and forcing with $\mathcal{Q}$ makes $\mathcal{P}(P)^V$ countable, then there is a $Q$-name $\tau$ for a $V$-generic filter for $P$ with the property that every condition in $P$ is forced by some condition in $Q$ to be in this filter. By Remark 4.1, then, there is a $P$-name $\tau$ for a partial order such that $P \ast \tau$ is forcing equivalent to $Q$.

By Remark 1.20, or the last part of Remark 1.13, if $P$ is a partial order in $V_\alpha$ (for $\alpha$ a limit ordinal greater than $\omega$), then forcing with $\mathcal{Q}_{<\alpha}$ makes $\mathcal{P}(P)^V$ countable, as does forcing with $\mathcal{P}_{<\alpha}$ below the condition $[\mathcal{P}(P)]^{\omega_0}$.

4.4 Exercise. Show that every set which is generic over $L$ exists (in a forcing extension) in a class sized model of $V=L$. While the model must be illfounded
if the set in question is not in $L$, the model can be made wellfounded up to any desired ordinal, though its $\omega_1$ will be illfounded if it is an generic ultrapower of $L$ via a partial order of the form $Q_{<\alpha}$.

The following theorem is a central part of Woodin’s theory of $\Omega$-logic. One interesting application of the theorem is the case where there exists exactly one huge cardinal or extendible cardinal or cardinal $\kappa$ which is $2^{\omega}$-supercompact, and $Q$ makes this cardinal countable.

**Theorem 4.5.** Suppose that $\delta$ is a Woodin cardinal, $\alpha < \delta$, $P$ and $Q$ are partial orders in $V_\delta$ and $P$ forces that $V_\alpha \models T$, for some theory $T$. Then there is a $Q$-name $\tau$ for a partial order such that, in the $Q\ast\tau$ extension there is an ordinal $\beta$ such that $V_\beta \models T$.

**Proof.** By Remark 4.3, there is a $Q$-name $\tau_0$ such that $Q \ast \tau_0$ is forcing-equivalent to the partial order $\mathbb{P}_{<\delta}$ below $[\mathcal{P}(Q)]^{\aleph_0}$. Let $G \subseteq \mathbb{P}_{<\delta}$ be a $V$-generic filter with $[\mathcal{P}(Q)]^{\aleph_0} \in G$, and let $j: V \rightarrow M$ be the corresponding embedding. Then $j(\delta) = \delta$, and, by Corollary 3.11, $V_\delta^M = V_\delta^{[G]}$. In $M$, $j(P)$ forces that $V_{j(\alpha)}$ models $T$. Since $\delta$ is strongly inaccessible in $V[G]$, in $V[G]$ also $j(P)$ forces that $V_{j(\alpha)}$ models $T$. \qed

**4.6 Example.** Suppose that $\gamma < \lambda < \kappa$ are regular cardinals below a Woodin cardinal $\kappa$. Forcing with $\mathbb{P}_{<\kappa}$ below the condition $b = \{ \alpha < \lambda \mid \text{cof}(\alpha) = \gamma \}$, we get that $j[\mathbb{b}] \in j(b)$, i.e., that in $M$, $j[\lambda]$ is an ordinal below $j(\lambda)$ of cofinality $j(\gamma)$. This means that the critical point of $j$ is $\lambda$, and that $\text{cof}(\lambda) = \gamma$ in $M$.

Furthermore, if $\alpha$ is such that $2^\alpha < \lambda$, then all subsets of $\alpha$ in $M$ are in $V$. Since $V_\alpha^M = V_\alpha^{[G]}$, these facts hold in $V[G]$ also.

**4.7 Example.** Suppose that $\kappa$ is a measurable cardinal, and let $\lambda > \kappa$ be a regular cardinal. Let $a$ be the set of $X < V_\kappa$ such that $\text{ot}(X \cap \kappa)$ and such that the transitive collapse of $X$ is constructible from a real. Then $a$ is stationary. To see this, fix a regular cardinal $\lambda > \kappa$ and a function $F: V_{<\lambda}^{<\omega} \rightarrow V_\kappa$. Let $Y = V_\lambda$ be countable with $\kappa$ and $F$ in $Y$, and let $U$ be a normal uniform ultrafilter on $\kappa$ in $Y$. Let $M$ be the transitive collapse of $X$. As in Theorem 1.20 from the first set of notes, we can successively produce $Y_\alpha$ ($\alpha < \kappa$) such that $Y_\alpha = Y$ and each $Y_{\alpha+1}$ has the form $Y_\alpha[\gamma]$ for some $\gamma \in \bigcap(Y_\alpha \cap U)$, taking unions at limits. Then $Y_\kappa$ will be an elementary submodel of $V_\kappa$ with $\text{ot}(Y_\kappa \cap \kappa) = \kappa$, and $Y_\kappa$ will be closed under $F$. Furthermore, letting $M_\alpha$ ($\alpha < \kappa$) be the transitive collapse of $Y_\alpha$, the sequence $\langle M_\alpha : \alpha < \kappa \rangle$ is the length $\kappa$-iteration of $M$ by the image of $U$ under the transitive collapse of $Y$. It follows that each $M_\alpha$ is in $L[M]$, including the transitive collapse of $Y_\kappa$.

Let $\delta > \lambda$ be a Woodin cardinal. Forcing with $\mathbb{P}_{<\delta}$ below $a$ then produces a real from which $V_\delta^M$ is constructible. The associated elementary embedding maps $\kappa$ to $\kappa$, so $\kappa$ is still measurable in $M$ and thus in $V[G]$.
5 Factoring

Lemma 5.1. Suppose that \( \delta \) is a Woodin cardinal, and \( \alpha > \delta \) is a limit ordinal. Let \( a \) be the set of \( X < V_{\delta+1} \) such that for every predense \( D \subseteq Q_{<\delta} \) in \( X \) there exists a \( d \in X \cap D \) with \( X \cap \bigcup d \in d \). Then \( a \) is stationary, and \( a \) is compatible with every element of \( Q_{<\delta} \). Furthermore, if \( G \subseteq P_{<\alpha} \) is a \( V \)-generic filter, then \( a \in G \) if and only if \( G \cap Q_{<\delta} \) is a \( V \)-generic filter for \( Q_{<\delta} \).

Proof. To show that \( a \) is stationary and compatible with every element of \( Q_{<\delta} \), fix a set \( b \in Q_{<\delta} \), a function \( F : V_{\delta+1}^{<\omega} \to V_{\delta+1} \), a regular cardinal \( \lambda > \delta \) and a countable \( X < V_\lambda \), with \( F \subseteq X \) and \( X \cap \bigcup b \in b \). Fix a bijection \( \pi : \omega \to \omega \times \omega \) such that \( \pi(i) \in (i+1)^2 \) for all \( i \in \omega \), and let \( \pi_0 \) and \( \pi_1 \) be functions on \( \omega \) such that \( \pi(n) = (\pi_0(n), \pi_1(n)) \) for all \( n \in \omega \). Recursively build a sequence \( \langle X_\alpha : \alpha \leq \omega \rangle \) of countable elementary submodels of \( V_\lambda \), a sequence \( \langle e_i : i < \omega \rangle \) and an increasing sequence of ordinals \( \langle \xi_i : i < \omega \rangle \) such that

1. \( X_0 = X \);
2. \( b \in V_{\xi_0} \);
3. for each \( i < \omega \), \( e_i \) is a surjection from \( \omega \) onto the predense subsets of \( Q_{<\delta} \) in \( X_i \);
4. for all \( i < j < \omega \), \( X_i \cap V_{\xi_i} = X_j \cap V_{\xi_i} \);
5. for each \( i < \omega \), for some \( d \in e_{\pi_0(i)}(\pi_1(i)) \), \( X_{\alpha+1} \cap \bigcup d \in d \);
6. \( X_\omega = \bigcup_{i<\omega} X_i \).

To achieve item (5), note that by Theorem 3.10, there exists in \( X_i \) an ordinal \( \xi_i \), greater than \( \xi_j \), for each \( j < i \), such that \( e_{\pi_0(i)}(\pi_1(i)) \cap P_{<\xi_i} \) is semi-proper.

For the last part of the theorem, suppose that \( G \subseteq P_{<\alpha} \) is a \( V \)-generic filter. Then \( a \in G \) if and only if \( j[A \cap j] \in j(a) \) if and only if \( j[V_{\delta+1}] \in j(a) \) if and only if, in \( M \), for each predense \( D \subseteq j(Q_{<\delta}) \) in \( j[V_{\delta+1}] \) there is a \( d \in D \cap j[V_{\delta+1}] \) with \( j[V_{\delta+1}] \cap \bigcup d \in d \) if and only if for each predense \( D \subseteq Q_{<\delta+1} \) there is a \( d \in D \) such that \( j[V_{\delta+1}] \cap \bigcup j(d) \in j(d) \). Finally, note that

\[
j[V_{\delta+1}] \cap \bigcup j(d) = j[V_{\delta+1}] \cap j(\bigcup d) = j(\bigcup d),
\]

and \( j[\bigcup d] \in d \) if and only if \( d \in G \).

\[\square\]

5.2 Remark. In the previous lemma, \( P_{<\alpha} \) can be replaced with a partial order of the form \( Q_{<\alpha} \).

5.3 Exercise. Suppose that \( \langle \delta_i : i < \omega \rangle \) is an increasing sequence of Woodin cardinals, with supremum \( \lambda \). Let \( a \) be the set of \( X < V_{\lambda+1} \) of cardinality less than \( \delta_0 \) such that \( \langle \delta_i : i < \omega \rangle \in X \), and for all \( i < \omega \) and every predense \( D \subseteq P_{<\delta_i} \) in \( X \) there exists a \( d \in X \cap D \) with \( X \cap \bigcup d \in d \). Show that \( a \) is stationary.
5.4 Remark. Suppose that $\alpha < \beta$ are limit ordinals greater than $\omega$, and that $G \subseteq \mathbb{P}_{<\alpha}$ is a $V$-generic filter such that $G \cap \mathbb{P}_{<\alpha}$ is $V$-generic for $\mathbb{P}_{<\alpha}$. Let $j_\alpha: V \to \text{Ult}(V, \mathbb{P}_{<\alpha})$ and $j_\beta: V \to \text{Ult}(V, G)$ be the corresponding elementary embeddings. Then the embedding $k: \text{Ult}(V, \mathbb{P}_{<\alpha}) \to \text{Ult}(V, G)$ defined by setting $k([f]_{\mathbb{P}_{<\alpha}^V}) = [f]_G$ is elementary, and has critical point at least $\alpha$. A similar fact holds for partial orders of the form $\mathbb{Q}^{\omega_1}_{<\beta}$.

6 Absoluteeness for the Chang Model

The Chang Model is $L(\text{Ord}^\omega)$, the constructible closure of the class of all countable sequences of ordinals. It is a model of ZF + DC, but not necessarily a model of the Axiom of Choice, as shown by Kunen. Solovay’s theorem on collapsing a strongly inaccessible cardinal applies to the Chang Model, showing the following.

Theorem 6.1 (Solovay). If $\kappa$ is a strongly inaccessible cardinal, then after forcing with $\text{Col}(\omega, <\kappa)$, every set of reals in the Chang Model is Lebesgue measurable, and satisfies the perfect set property and the property of Baire.

Theorem 6.2. Suppose that $\delta$ is a Woodin limit of Woodin cardinals. Then in a forcing extension there is an elementary embedding from the Chang Model of $V$ to the Chang Model of a forcing extension of $V$ by $\text{Col}(\omega, <\delta)$.

Proof. Let $G \subseteq \mathbb{Q}_{<\delta}$ be a $V$-generic filter, and let $j: V \to M$ be the corresponding elementary embedding. Then $M$ and $V[G]$ have the same Chang Model, and $j$ maps the Chang Model of $V$ to the Chang Model of $M$. Working in $V[G]$, let $\mathbb{P}$ be the partial order consisting of $V$-generic filters for partial orders of the form $\text{Col}(\omega, <\alpha)$, for some $\alpha < \delta$, ordered by extension. Since $\delta = \omega_1^{V[G]}$ and $\delta$ is strongly inaccessible in $V$, there exist such generic filters for each such $\alpha$.

Let $H$ be $V[G]$-generic for $\mathbb{P}$. It suffices to see that $H$ is $V$-generic for $\text{Col}(\omega, <\delta)$, and that $V[G]$ and $V[H]$ have the same Chang Model. That $H$ is $V$-generic for $\text{Col}(\omega, <\delta)$ follows from the fact that if $D \subseteq \text{Col}(\omega, <\delta)$ is predense and $h \subseteq \text{Col}(\omega, <\alpha)$ in $V$-generic, for some $\alpha < \delta$, then $p \cap \text{Col}(\omega, <\alpha) \in h$ for some $p \in D$. Fixing $\beta < \delta$ such that $p \in \text{Col}(\omega, <\beta)$, we have that $\text{Col}(\omega, <\beta)$ is isomorphic to $\text{Col}(\omega, <\alpha) \times \text{Col}(\omega, <\beta)$, which means that there is a $V$-generic filter $h' \subseteq \text{Col}(\omega, <\beta)$ extending $h$ with $p \in h'$. By genericity, then, $H \cap D$ is nonempty.

To see that $V[G]$ and $V[H]$ have the same Chang Model, note first of all that $\delta = \omega_1^{V[H]}$, which means that every countable set of ordinals in $V[H]$ is in $V[H \cap \text{Col}(\omega, \alpha)]$ for some $\alpha < \delta$, and thus in $V[G]$. For the reverse inclusion, suppose that $x$ is a countable set of ordinals in $V[G]$, and that $h \in V[G]$ is $V$-generic for $\text{Col}(\omega, <\alpha)$, for some $\alpha < \delta$. By Lemma 5.1, and the fact that $\delta = \omega_1^{V[G]}$, we may fix a $\beta < \delta$ such that $h$ and $x$ are in $V[G \cap \mathbb{Q}_{<\beta}]$. By Remark 4.1, there is a $\text{Col}(\omega, <\alpha)$-name $\tau$ such that $\text{Col}(\omega, <\alpha) \star \tau$ is forcing equivalent to $\mathbb{Q}_{<\beta}$. Moreover, using a fixed $\mathbb{Q}_{<\beta}$-name $\rho$ for which $h = \rho \cap \mathbb{Q}_{<\beta}$, we may (applying the proof of Remark 4.1) choose $\tau$ so that $V[G \cap \mathbb{Q}_{<\beta}]$ is a generic
extension of $V[h]$ via $\tau$. Furthermore, there exist an ordinal $\gamma < \delta$ and a $\tau$-name $\sigma \in V[h]$ such that, in $V[h]$, $\tau * \sigma$ is forcing equivalent to $\text{Col}(\omega, <[\alpha, \gamma])$. As $(2^{\gamma})^{V[h]}$ is countable in $V[G]$, there exists in $V[G]$ a $V[h]$-generic filter $h' \subseteq \text{Col}(\omega, <[\alpha, \gamma])$ such that $G \cap Q_{<\beta} \in V[h, h']$. \qed

The following classical theorem is due to McAloon.

6.3 Exercise. Show that if $P$ is a partial order such that forcing with $P$ makes $\tau$ countable, then $P$ is forcing-equivalent to $\text{Col}(\omega, P)$. (Hint : Fix a $P$-name $\pi$ for a bijection between $\omega$ and the generic filter. Recursively build a function $\pi : \text{Col}(\omega, P) \to P$ in such a way that $\pi(\emptyset) = 1_P$, and, for all $a \in \text{Col}(\omega, P)$, $\pi\{a \cup \{a, p\} : p \in P\}$ is a maximal antichain below $\pi(a)$ and $\pi(a)$ decides $\tau \upharpoonright a$. To see that the range of $\pi$ is dense in $P$, fix $p \in P$ and a condition $p' \leq p$ forcing that $\tau(p) = n$, for some $n \in \omega$. Fix $a \in \text{Col}(\omega, P)$ of length greater than $n$ such that $\pi(a)$ and $p'$ are compatible, and show that $\pi(a) \leq p$.)

6.4 Remark. The previous exercise shows that for any cardinal $\kappa$, and any partial order $P$ of cardinality less than $\kappa$, $P \times \text{Col}(\omega, <\kappa)$ is forcing-equivalent to $\text{Col}(\omega, <\kappa)$ (as for any $\gamma > |P|$, $\text{Col}(\omega, <\gamma)$ can be replaced with $P \times \text{Col}(\omega, <\gamma)$). Combined with Theorem 6.2, this shows that if $\delta$ is a Woodin limit of Woodin cardinals, than no forcing of cardinality less than $\delta$ can change the theory of the Chang Model. The large cardinal hypothesis required for this result is much weaker. It suffices to assume that $\delta$ is a limit of Woodin cardinals, and that there is a measurable cardinal above $\delta$, and even this can be weakened.

6.5 Exercise. Suppose that $\{\delta_i : \alpha \leq \omega\}$ are Woodin cardinals, listed in increasing order, and let $\lambda = \sup\{\delta_i : i < \omega\}$. Show that the $L(\mathbb{R})$ of $V$ is elementarily equivalent to a model of the form $L(\mathbb{R}^*)$ in a forcing extension of $V$ by $\text{Col}(\omega, <\lambda)$, where $\mathbb{R}^*$ is the set of reals existing in models of the form $V[G \cap \text{Col}(\omega, <\gamma)]$ for some $\gamma < \lambda$, where $G$ is the generic filter for $\text{Col}(\omega, <\gamma)$. (Hint : Force with $Q_{<\lambda}$ below the condition $a$ from Exercise 5.3. For each $i < \omega$, let $j_i : V \to M_i$ be the embedding induced by $G \cap Q_{<\delta_i}$. Let $N$ be the direct limit of the models $M_i$ ($i \in \omega$) via the factor embeddings defined in Remark 5.4. Show that $N$ embeds into $M_\omega$, and is therefore wellfounded. Now mimic the proof of Theorem 6.2 to show that the $L(\mathbb{R})$ of $N$ is the same as a model of the form $L(\mathbb{R}^*)$ of a forcing extension of $V$ by $\text{Col}(\omega, <\lambda)$.)

7 $\mathbb{R}^#$

Our next goal is to sketch a proof of the fact that, assuming sufficiently many Woodin cardinals, every set of reals is weakly homogeneously Suslin. This will be useful in the development of $\mathbb{P}_{\text{max}}$. We will sketch the relationship between this fact and Woodin’s theorem that the existence of infinitely many Woodin cardinals below a measurable implies that the Axiom of Determinacy holds in $L(\mathbb{R})$. This will involve some black boxes, however. The first of these is the set $\mathbb{R}^#$. 15
Very briefly, given a set of reals $A$ such that $L(A) \cap \mathbb{R} = A$, $A^#$ is a complete, consistent theory in the language of set theory expanded by adding constant symbols $c_x$ for each $x \in A$ and constant symbols $i_n$ ($n \in \omega$) which represent ordinal indiscernibles. This theory in effect gives a recipe which for any ordinal $\alpha$ builds a model $\Gamma(A^#, \alpha)$ of $ZF + V=L(\mathbb{R})$ whose reals are exactly $A$, where we have $\alpha$ many ordinals playing the role of the indiscernibles. Note that in $L(\mathbb{R})$, every set is definable from a finite set of reals and ordinals, and for each fixed finite set of reals $a$ the class of sets ordinal definable from $a$ has a definable wellordering. In the models $\Gamma(A^#, \alpha)$, every set is definable from a finite set indiscernibles and elements of $A$, and the theory $A^#$ explicitly defines the relations $\in$ and $=$ on these terms.

Let us black box the following facts about $A^#$.

- If $U$ is a normal uniform measure on a cardinal $\kappa$, then there is a set $X \subseteq U$ such that for each $n \in \omega$, any two increasing $n$-tuples from $X$ satisfy the same formulas in $L(\mathbb{R})$, allowing constants for real numbers. The theory satisfied by the finite tuples from $X$ is $\mathbb{R}^#$ (in fact, a completely Jónsson cardinal is enough).
- There is a $\Delta_0$ formula $\phi$ such that, for any set of reals $A$ for which $A = \mathbb{R} \cap L(A)$, $A^#$ is the unique set $B \subseteq A$ such that
  
  - $(A, \{B\}, \in) \models \phi(B)$;
  - for all countable ordinals $\alpha$, $\Gamma(B, \alpha)$ is wellfounded.
- In each model $\Gamma(A^#, \alpha)$, the indiscernibles form a a club of ordertype $\alpha$, and for all cardinals $\kappa > |A|$, the ordinal height of $\Gamma(\mathbb{R}^#, \kappa)$ is $\kappa$, and $\kappa$ is an indiscernible of $\Gamma(A^#, \alpha)$ for all $\alpha > \kappa$.
- Every set of real in $L(A)$ is definable from $A^#$ and a finite sequence of elements of $A$.

Since $\mathbb{R}^#$ is a definable element of the Chang model, we have that if $\delta$ is a Woodin limit of Woodin cardinals, if $V[G]$ is any generic extension via a partial order of cardinality less than $\delta$, then $(\mathbb{R}^#)^V \subseteq (\mathbb{R}^#)^{V[G]}$.

Now suppose that $\lambda$ is the limit of an increasing sequence of Woodin cardinals $\langle \delta_i : i < \omega \rangle$, and that $\alpha > \lambda$ is a limit ordinal. Let $a$ be the set of countable $X \times V_{\lambda+1}$ such that for all $i < \omega$, and every predense $D \subseteq Q_{\delta_i}$, in $X$, there is a $d \in X \cap D$ such that $X \cap d \in d$. Then $a \in P_{<\alpha}$, and $a$ forces in $P_{<\alpha}$ that $G \cap Q_{<\delta_i}$ is $V$-generic for each $i < \omega$.

The generic filter $G$ then gives a sequence of embeddings $j_i : V \rightarrow M_i$, each induced by $G \cap Q_{<\delta_i}$, with factor embeddings and limit model $N$. Adapting the argument above for the Chang Model, one can force over $V[G]$ to find a $V$-generic $H \subseteq \text{Col}(\omega, \lambda)$ such that $R^N = \bigcup\{R^{V[H \cap \text{Col}(\omega, \alpha)]} : \alpha < \lambda\}$. Assuming enough wellfoundedness for $N$ (which holds if $\alpha$ is Woodin or even completely Jónsson or less), we get that $(\mathbb{R}^#)^N = (\mathbb{R}^N)^#$. Again, this gives us that whenever $V[G]$ is any generic extension via a partial order of cardinality less than $\lambda$, then $(\mathbb{R}^#)^V \subseteq (\mathbb{R}^#)^{V[G]}$. 

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8 Towers of measures

8.1 Definition. A tower of measures is a sequence \( \langle \mu_i : i \in \omega \rangle \) such that, for some fixed set \( Z \),

- each \( \mu_i \) is an ultrafilter on \( Z^i \);
- for all \( i < j < \omega \) and all \( A \subseteq Z^i, A \in \mu_i \) if and only if
  \[ \{ \sigma \in Z^i \mid \sigma \upharpoonright i \in A \} \in \mu_i. \]

Note that \( \mu_0 \) is always the trivial ultrafilter on \( \emptyset \).

A tower of measures \( \bar{\mu} = \langle \mu_i : i < \omega \rangle \) gives rise to a sequence of elementary embeddings \( j_i : V \rightarrow \text{Ult}(V, \mu_i) \), for each \( i_0 < i_1 < \omega \) there is an elementary factor embedding \( k_{i_0,i_1} : \text{Ult}(V, \mu_{i_0}) \rightarrow \text{Ult}(V, \mu_{i_1}) \) defined by setting \( k_{i_0,i_1}([f]_{\mu_{i_0}}) = [f']_{\mu_{i_1}} \), where \( f'(\sigma) = f(\sigma^{i_0}) \) for all \( \sigma \in Z^{i_1} \) (and \( Z \) is such that each \( \mu_i \) concentrates on \( Z^i \)).

The direct limit of the embeddings \( k_{i_0,i_1} \) gives rise to a limit model \( \text{Ult}(V, \bar{\mu}) \), whose elements are represented by functions \( f : Z^i \rightarrow V \), for some \( i < \omega \). For a relation \( R \in \{ \in, = \} \), we set \([f]_{\bar{\mu}} R [g]_{\bar{\mu}}\) if and only if
\[
\{ \sigma \in Z^{i_2} \mid f(\sigma^{i_0}) R g(\sigma^{i_1}) \} \in \mu_{i_2},
\]
for \( f : Z^{i_0} \rightarrow V \), \( g : Z^{i_1} \rightarrow V \) and \( i_2 = \max\{i_0, i_1\} \).

8.2 Definition. A tower of measures \( \langle \mu_i : i < \omega \rangle \) is said to be countably complete if, for each sequence \( \langle A_i : i \in \omega \rangle \) such that each \( A \in \mu_i \), there is a sequence \( \sigma \in \omega_\mu \) such that \( \sigma \upharpoonright i \in A_i \) for all \( i \in \omega \). We call such a \( \sigma \) a thread through \( \langle A_i : i < \omega \rangle \).

8.3 Exercise. Suppose that \( \langle \mu_i : i < \omega \rangle \) is a countably complete tower of measures. Show that each \( \mu_i \) is countably complete, which implies that \( \text{Ult}(V, \mu_i) \) is wellfounded. Show furthermore that \( \text{Ult}(V, \bar{\mu}) \) is wellfounded.

Lemma 8.4. If \( \langle \mu_i : i < \omega \rangle \) is a tower of measures which is not countably complete, then \( \text{Ult}(V, \bar{\mu}) \) is not wellfounded

Proof. By Exercise 1.9 from the first set of notes, a failure of countable completeness for any \( \mu_i \) gives that \( \text{Ult}(V, \mu_i) \) is illfounded, which implies that \( \text{Ult}(V, \bar{\mu}) \) is illfounded since \( \text{Ult}(V, \mu_i) \) embeds into it. So assume that each \( \mu_i \) is countably complete and fix \( \langle A_i : i \in \omega \rangle \) such that each \( A_i \in \mu_i \) but there is not thread through \( \langle A_i : i < \omega \rangle \). Replacing each \( A_i \) with \( \bigcap_{n \in \omega} \{ \sigma \upharpoonright i \mid \sigma \in A_n \} \) be may assume that for all \( i_0 < i_1 < \omega \) and all \( \sigma \in A_{i_1}, \sigma^{i_0} \in A_{i_0} \). Then \( T = \bigcup_{i \in \omega} A_i \) is a wellfounded tree, and there is a function \( f \) on \( T \) defined by letting each \( f(\sigma) \) be the least ordinal greater than \( f(\tau) \) for all \( \tau < T \) properly extending \( \sigma \) (i.e., a rank function on \( T \)). For each \( i < \omega \), let \( g_i : Z^i \rightarrow V \) (where each \( \mu_i \) is an ultrafilter on \( Z^i \)) be such that \( g_i(\sigma) = f(\sigma) \) for all \( \sigma \in A_i \). Then the functions \( g_i \) \((i \in \omega)\) represent a descending \( \omega \)-sequence in \( \text{Ult}(V, \bar{\mu}) \).

\[ \square \]
8.5 Definition. Given a cardinal \( \kappa \), a set \( A \subseteq \omega^\omega \) is \( \kappa \)-homogeneously Suslin if, for some set \( Z \), there is a set \( \{ \mu_s \mid s \in \omega^{<\omega} \} \) such that

- each \( \mu_s \) is a \( \kappa \)-complete ultrafilter on \( Z^{[s]} \);
- \( A \) is the set of \( x \in \omega^\omega \) for which \( \langle \mu_x^{[i]} : i < \omega \rangle \) is a countably complete tower.

We say that \( A \) is homogeneously Suslin if it is \( \aleph_1 \)-homogeneously Suslin. We say that \( A \) is \( <\kappa \)-homogeneously Suslin if it is \( \gamma \)-homogeneously Suslin for all \( \gamma < \kappa \).

8.6 Definition. Given a cardinal \( \kappa \), a set \( A \subseteq \omega^\omega \) is \( \kappa \)-weakly homogeneously Suslin if, for some set \( Z \), there is a set \( \{ \mu_{s,t} \mid s,t \in \omega^{<\omega} \} \) such that

- each \( \mu_{s,t} \) is a \( \kappa \)-complete ultrafilter on \( Z^{[s]} \);
- \( A \) is the set of \( x \in \omega^\omega \) for which there exists a \( y \in \omega^\omega \) such that \( \langle \mu_x^{[i,y]} : i < \omega \rangle \) is a countably complete tower.

We say that \( A \) is weakly homogeneously Suslin if it is \( \aleph_1 \)-homogeneously Suslin. We say that \( A \) is \( <\kappa \)-weakly homogeneously Suslin if it is \( \gamma \)-homogeneously Suslin for all \( \gamma < \kappa \).

We will black box the following facts, which together give the relationship between Woodin cardinals and determinacy in the projective hierarchy (i.e., that the existence of \( n \) Woodin cardinals below a measurable cardinal implies that all \( \Pi^1_n \) sets are determined). The Martin-Steel theorem also shows that if \( \lambda \) is a limit of Woodin cardinals, then the \( <\lambda \)-weakly homogeneously Suslin sets are exactly the \( <\lambda \)-homogeneously Suslin sets.

Theorem 8.7 (Martin). If \( \kappa \) is a measurable cardinal, then coanalytic sets are \( \kappa \)-homogeneously Suslin.

Theorem 8.8 (Martin). Homogeneously Suslin subsets of \( \omega^\omega \) are determined.

Theorem 8.9 (Martin-Steel). If \( \delta \) is a Woodin cardinal and \( A \subseteq \omega^\omega \) is weakly homogeneously Suslin, then \( \omega^\omega \setminus A \) is \( <\delta \)-homogeneously Suslin.

8.10 Definition. For our purposes, a tree on a set \( X \) is a set of finite sequences from \( X \), closed under initial segments. Given a set \( X \) and a tree \( T \) on \( \omega \times X \), the projection of \( T \), \( p[T] \), is the set of \( x \in \omega^\omega \) such that for some \( y \in X^\omega \), \( (x|n, y|n) \in T \) for all \( n \in \omega \).

8.11 Definition. Given a cardinal \( \kappa \), a set \( A \subseteq \omega^\omega \) is \( \kappa \)-universally Baire if for some ordinal \( \gamma \) there are trees \( S \) and \( T \) on \( \omega \times \gamma \) such that \( A = p[S] \) and, in all forcing extensions by partial orders of cardinality at most \( \kappa \), \( p[S] = \omega^\omega \setminus p[T] \). We say that \( A \) is \( <\kappa \)-universally Baire if it is \( \lambda \)-universally Baire for all \( \lambda < \kappa \), and universally Baire if it is \( \kappa \)-universally Baire for all cardinals \( \kappa \).
8.12 Remark. By McAloon’s result, the definition of \( \kappa \)-universally Baire does not change if one replaces “by partial orders of cardinality at most \( \kappa \)” with \( \text{Col}(\omega, \kappa) \).

The next two theorems show that when \( \lambda \) is a limit of Woodin cardinals, the \(<\lambda\)-weakly homogeneously Suslin subsets of \( \omega^\omega \) are exactly the \(<\lambda\)-universally Baire sets. The second of these theorems uses the stationary tower.

Theorem 8.13 (Martin-Solovay). If \( \kappa \) is a cardinal, and \( A \subseteq \omega^\omega \) is \( \kappa \)-weakly homogeneously Suslin, then \( A \) is \(<\kappa\)-universally Baire.

Theorem 8.14 (Woodin). Suppose that \( \delta \) is a Woodin cardinal and \( S, T \) are trees on \( \omega \times \gamma \), for some ordinal \( \gamma \) such that \( S \) and \( T \) project to complements in all generic extensions via \( \text{Col}(\omega, \delta) \). Then \( p[S] \) is \(<\delta\)-weakly homogeneously Suslin.

The following theorem is known as the Tree Production Lemma. It is our primary means of showing that sets of reals are universally Baire. Note that whenever \( r \) is subset of \( V \) which exists in a set generic extension of \( V \), \( V[r] \) is also a set-generic extension of \( V \) (via the complete Boolean subalgebra generated by the terms \( [x \in \tau] \), for each \( x \) in some superset of \( r \) in \( V \), and a name \( \tau \) giving rise to \( r \)). The proof of the Lemma uses the stationary tower.

Theorem 8.15 (Woodin). Suppose that \( \delta \) is a Woodin cardinal. Let \( \phi \) and \( \psi \) be binary formulas, let \( x \) and \( y \) be sets, and assume that the empty condition in \( Q_{<\delta} \) forces that for each real number \( r \),

\[
M \models \phi(r, j(y)) \iff V[r] \models \psi(r, x),
\]

where \( j: V \to M \) is the induced embedding. Then there exist trees \( S \) and \( T \) on \( \omega \times \gamma \), for some ordinal \( \gamma \) such that \( p[S] = \{ r \in R \mid \phi(r, y) \} \) and \( S \) and \( T \) project to complements in any forcing extension via \( \text{Col}(\omega, <\delta) \).

Let us consider the Tree Production Lemma in the context where there exist \( \omega + 1 \) many Woodin cardinals above \( \delta \) (or just \( \omega \) many plus a measurable cardinal). Let \( \lambda \) be the limit of the first \( \omega \) Woodin cardinals above \( \delta \). Let \( \phi(y) \) be the formula “\( r \in R^\# \)”, and let \( \psi \) be the formula “\( r \in (R^*)^\# \) after forcing with \( \text{Col}(\omega, \lambda) \)”. The arguments given above show that the hypotheses of the Tree Production Lemma are satisfied, and thus that \( (R^\#)^V \) is \(<\delta\)-universally Baire. Since we could apply the same argument for each Woodin cardinal below \( \lambda \), we have that \( (R^\#)^V \) is \(<\delta\)-universally Baire. Since every set of reals is a continuous preimage (in some sense just a projection) of \( R^\# \), we have that each such set is \(<\delta\)-universally Baire, and thus \(<\delta\)-homogeneously Suslin.

The Tree Production Lemma is also used to show that if \( \lambda \) is a limit of Woodin cardinals, then the \( \text{Col}(\omega, <\lambda) \)-extension contains inner models satisfying determinacy.